Games on Multiplex Networks*

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Abstract

We provide a simple network model of multiple layers, an important but understudied topic in the network literature. On any layer, agents' incentives are influenced by their within-layer social ties. Facing aggregate effort constraints, agents optimize across layers, which may have heterogeneous network structures. We first characterize the equilibrium of this game and determine the importance of both within and between-layer interactions in terms of shock propagation. Then, we identify the optimal targeting interventions with multiplexity in which the planner needs to take into account both the impact of its policy on one layer and that on the other interconnected layers. Applications and simulations to the management of multiple social relationships and multiple public goods help us understand the complex mechanisms behind our results.

Keywords: multiplexity, social network, targeting interventions, policies

JEL classifications: L14, L13, D43, D85

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1 Introduction

Motivation

Multiplex networks are complex systems that consist of multiple layers or networks of interconnected nodes. Each layer can represent a different type of relationship or interaction between the same set of nodes. For example, in a social network, we could have one layer representing friendships between individuals, another layer representing professional connections, and a third layer representing romantic relationships. In the case of online social networks, the different layers could represent the different types of interactions that take place on a social media platform, such as Facebook or Twitter. For instance, one layer could represent the friendship between users, while another layer could represent the interactions between users based on shared interests or groups. Multiplex networks can also be used to study the professional relationships between individuals, such as collaborations between researchers or business partnerships. Different layers could represent different types of collaborations, such as joint publications or shared patents.

A key feature of multiplex or multilayer networks is that they allow us to capture the complexity and richness of real-world systems. In many cases, a single network or layer is not enough to fully capture the different types of relationships that exist between the nodes.[1] By explicitly modeling and analyzing multiple layers, we can better understand the structure of the system as a whole and the underlying mechanism at play.

In this paper, we propose the first network-games model on multiplex networks in which each agent has to decide how much effort is required in each layer. Indeed, given a layer-specific value, each agent makes effort under either strategic complementarities or strategic substitutes. Agents are heterogeneous in terms of both their endowments of resources and their network positions in different layers. Moreover, agents have to take into account the "connection" between layers; for example, if we interpret the different layers as social connections such as family, professional, romantic, and religious networks, each agent needs to take into account the fixed time/attention allocation across these different layers. For instance, if efforts are

[1] For example, Chandrasekhar et al. (2024), who studied the importance of multilayer networks in diffusion, showed that some layers are more predictive than others. In particular, the advice, kero/rice/money, and backbone networks are more predictive of diffusion of information/behavior than others such as the socialize, temple, or decision help network.

strategic complements, spillovers are large, and the player, who is central in one network/layer, may exert considerable effort in this network but then, given their time constraint, may reduce their effort in the other layers.

Main results

First, despite the high dimensionality of the strategy space and the heterogeneity of social links across layers, we show that our multiplexity network game has a best-reply potential. In particular, we make our model tractable by employing a methodology called sign equivalent transformation (SET) (Zenou and Zhou, 2022), which transforms the equilibrium first-order conditions of the original game into optimality conditions of maximizing a single function, called a best-reply potential. By exploiting this approach, we provide conditions based on the spectral properites of each graph/network to show the existence and uniqueness of a Nash equilibrium in our game.

Second, we explicitly characterize our multilayer network equilibrium. We show that, at any interior equilibrium, for each layer s, the effort of each agent i depends on a layer-specific value α^s and their weighted Katz-Bonacich centrality, whose weight is equal to the *multiplexity influence* parameter $\mu_i^* = 1/\lambda_i^*$, where λ_i^* is the multiplier of i's time/budget constraint. Importantly, the multiplier λ_i^* is endogenously determined in equilibrium by the agent's resource constraints and the interactions within and across layers. We call μ_i^* the system-level centrality as it summarizes an agent's social influence in all layers and its budget resource. We provide an explicit expression of the multiplexity influence and show that higher system-level centrality μ_i^* leads to higher equilibrium utility U_i^*.

Third, we provide two main applications throughout the paper to illustrate our main results. In the first one, we consider the management of multiple social relationships in which each agent has to allocate their time/attention across different layers of social interactions such as their family network, professional network, and religious network. In the second application, we study multiple public goods in which the first good is a local public good (any network), the second one is a pure private good (the network is empty), and the third good is a global pure public good (the network is complete).[2] In this case, contrary to the monolayer case (Bramoullé and Kranton, 2007; Allouch, 2015), we show that, in equilibrium, all agents obtain the

[2]It is straightforward to extend this model to any finite number of local public goods.

same utility and consume the same amount of private and global public good. This is because we add one layer in which the network is complete due to the nature of the global public good, which induces all agents to consume the same amount of private good and to obtain the same utility. We also show that a budget-balanced policy that involves giving a transfer to each agent is *neutral* in the sense that, independently to whom the transfer is given, it leaves unchanged the consumption of the private and public goods of each consumer.

Fourth, when we restrict our analysis to the case of the same network structure between different layers, we obtain sharper results in terms of equilibrium characterization. In this case, we extend the principle component decomposition technique of Galeotti et al. (2020) and Chen et al. (2022) to derive simpler equilibrium characterizations in our multilayer setting. In particular, we show that different agents act as if they were independent of each other. We can also derive simpler expressions for the planner's targeted problems.

Fifth, we consider *regular multiplex networks* in which, for each layer/network s, the network is regular with degree d^s. We obtain a clean characterization of the agent's effort in each layer, which only depends on the layer-specific preference weight α^s, and the scalar social multiplier in each network. We also find that the effort in layer s increases with α_s but decreases with α_t, for $t \neq s$; increases (decreases) with the degree d^s in layer s when efforts are strategic complements (substitutes); and decreases (increases) with degree d^t, for $t \neq s$, when efforts are strategic complements (substitutes).

Sixth, we determine the inefficiency of the equilibrium allocations—that is, the gap between equilibrium and first-best allocations. The equilibrium is inefficient because each agent i does not take into account the effects of their effort x_i^s on other agents' payoff in their layer as well as in other layers. When efforts are strategic complements (substitutes), agent i overestimates (underestimates) the marginal welfare effects of x_i^s. Since the total time budget is fixed for each player i, the discrepancy between the equilibrium and first best boils down to the relative allocations across different layers. If we consider regular networks, this discrepancy mostly depends on the difference in the social multiplier effects between the layers.

Seventh, we explore the comparative statics results and policy implications of our model, in particular, how an exogenous shock (such as the initial endowment of each agent or the individual income) propagates within and between layers. We show that there is a direct negative effect of increasing the initial endowment (which is layer-specific) on individual efforts in a given layer since endowment and efforts are substitutes. There is also an indirect effect since each individual must reallocate their efforts on the other layers, which could be positive or negative depending on whether the efforts are strategic substitutes or complements and on the weight each agent puts on each layer. The effect of an increase in individual endowment and income on utility and welfare depends on whether we assume positive or negative spillovers between players. We also show that when the individual income (which is not layer specific) increases, agents have more resources and thus can allocate their efforts differently across layers. In that case, utility and welfare always increase. We illustrate these results using numerical simulations for the two applications mentioned above and consider three cases: (i) the management of multiple social relationships with strategic complements and negative spillovers, (ii) the management of multiple social relationships with strategic substitutes and positive spillovers, (iii) multiple public goods with one local public good (any network with strategic substitutes and positive spillovers), one private good (empty network), and one global public good (complete network with strategic substitutes and positive spillovers).

Finally, we revisit the standard targeting intervention problem analyzed in monolayer networks (Ballester et al., 2006; Galeotti et al., 2020; Kor and Zhou, 2022) to the case of multilayer networks. We analyze either a subsidy/taxation on the endowment or an income policy that determines which player to target to maximize welfare. Importantly, the planner now needs to take into account the effect on both the current layer and the other layers. For each policy, we provide an index formula that depends on both the matrix of interactions between agents in a given layer and the interaction between layers. We illustrate our results by running simulations using the three cases mentioned above.

Related literature

Our paper is related to different strands of the literature.

Network games on a single-layer network

In the literature on network games, most papers assume a single-layer network (e.g., Ballester et al., 2006; Bramoullé and Kranton, 2007; Bramoullé et al., 2014).[3]

[3] For an overview, see Jackson and Zenou (2015) and Bramoullé and Kranton (2016).

Chen, Zenou, and Zhou (2018) were among the first to extend Ballester, Calvó-Armengol, and Zenou (2006) study to analyze multiple interdependent activities among users embedded on a single network. In this paper, we introduce a novel class of games played on multiple layers of social links. Despite the high dimensionality of strategy space and heterogeneity of social links across layers, we are able to characterize the Nash equilibrium, and show that it exists and find conditions under which it is unique. We also derive interesting comparative statics results and policy implications. We believe we are the first to propose a general model on games on multiplex networks.

Multilayer networks

The study of multilayer networks is growing.[4] Cheng, Huang, and Xing (2021) proposed a theory of multiplexity and showed how to sustain cooperation with multiple social relations. Billand et al. (2021) developed a network formation model on multilayer networks, while Joshi et al. (2020) investigated the partial formation of a strategically interacting two-dimensional multiplex in the framework of a linear quadratic game. Joshi et al. (2023) examined how an initial seed layer influences link formation on other layers and shapes their equilibrium topology. Finally, Tzavellas (2023) focused on macroeconomic issues by extending the monolayer network concepts of systemic importance and microinduced aggregate fluctuations to their multilayered counterparts. It is shown that idiosyncratic shocks can cascade in the system through both intra- and inter-network margins.[5]

In the current paper, we take a different approach by not focusing on network formation but rather take the multilayer networks as given and analyze the allocation of efforts within and between different layers. In addition to establishing general existence and uniqueness results, we also illustrate our results with different applications in terms of comparative statics and targeting.

[4]For a comprehensive survey from the perspective of complex networks, see Kivelä et al. (2014).

[5]There is an interesting literature in finance that constructs multilayer structures of different financial systems and markets and investigates their consequences on systemic risk (see e.g., Poledna et al., 2015; Aldasoro and Alves, 2018; Bardoscia et al., 2019).

2 Model setup

2.1 Preferences

Consider a simultaneous-move multilayer network game. Let $\mathcal{N} = \{1, \cdots, n\}$ denote the set of players and $S = \{1, \cdots, s\}$ the set of layers. Assume $n \geq 2, s \geq 1$. Each agent i's strategy is a vector $\mathbf{x}_i = (x_i^1, \cdots, x_i^s) \in K_i := \{\mathbf{x}_i \in \mathbf{R}_+^s \mid \sum_{s \in \mathcal{S}} x_i^s = T_i\}$, where x_i^s denotes agent i's action on layer $s \in \mathcal{S}$, and $T_i > 0$ is a cap on the aggregate action of i; for example, T_i can be interpreted as a budget or time constraint for agent i.

Given $\mathbf{x}_{-i} = (\mathbf{x}_1, \cdots, \mathbf{x}_{i-1}, \mathbf{x}_{i+1}, \cdots, \mathbf{x}_n) \in K_{-i} = \prod_{j \neq i} K_j$, the utility of agent i from taking action $\mathbf{x}_i \in K_i$ is equal to [6]

$$U_i(\mathbf{x}_i, \mathbf{x}_{-i}) = \sum_{s \in \mathcal{S}} \alpha^s \ln \left(v_i^s + x_i^s + \phi^s \sum_{j \in \mathcal{N}} g_{ij}^s x_j^s \right), \tag{1}$$

where the parameter $\alpha^s > 0$ represents the preference weight of layer s, $\phi^s \in \mathbf{R}$ denotes the network spillover parameter on layer s, and g_{ij}^s is the social tie between i and j on layer s. The parameter $v_i^s \in \mathbf{R}$ is a constant and captures the idiosyncratic heterogeneous endowment of each agent i, while $\phi^s \sum_{j \in \mathcal{N}} g_{ij}^s x_j^s$ represents the impact of the actions of the players linked to j on i's utility.

We are interested in pure strategy Nash equilibrium (or simply equilibrium) of this simultaneous-move game.

Before solving the model, we briefly discuss several features of the model setup.

1. For each layer $s \in \mathcal{S}$, we let $\mathbf{G}^s = (g_{ij}^s)_{1 \leq i,j \leq s}$ denote the adjacency matrix. Following the literature, we assume that $\mathbf{G}^s$ is undirected with no self-loops, i.e., $g_{ij}^s = g_{ji}^s \geq 0, \forall i, j \in N$ and $g_{ii}^s = 0, \forall i \in \mathcal{N}$. The social network could vary across layers. An agent can be central in a layer s' while being peripheral on another layer s''.

2. The parameter $\phi^s \in \mathbf{R}$ can be positive or negative depending on the economic

[6] Alternatively, we can assume $\tilde{U}_i(\mathbf{x}_i, \mathbf{x}_{-i}) = \prod_{s \in \mathcal{S}} \left(v_i^s + x_i^s + \phi^s \sum_{j \in \mathcal{N}} g_{ij}^s x_j^s \right)^{\alpha^s}$, which is the exponential of $U_i(\mathbf{x}_i, \mathbf{x}_{-i})$ defined in (1). Clearly, under both specifications, players' best reply functions and equilibrium efforts must be the same. However, some welfare results may be different. To fix ideas, we adopt the specification in (1) throughout the paper.

contexts. Direct computation reveals that

$$sign\left\{\frac{\partial U_i}{\partial x_j^s}\right\} = sign\left\{\phi^s \times g_{ij}^s\right\}, \forall i, j \in \mathcal{N}, \tag{2}$$

and

$$sign\left\{\frac{\partial^2 U_i}{\partial x_i^s \partial x_j^s}\right\} = -sign\left\{\phi^s \times g_{ij}^s\right\}, \forall i, j \in \mathcal{N}. \tag{3}$$

Consequently, on a fixed layer s, when $\phi^s < 0$, the efforts of two socially connected players i, j are *strategic complements* but player j's effort exerts a *negative externality* on i's utility; in contrast, when $\phi^s > 0$, the two efforts are *strategic substitutes* but player j's effort exerts a *positive externality* on i's utility.

3. The setting of a positive ϕ^s, that is, efforts are strategic substitutes and neighbors exert positive externalities on each other, arises quite naturally in models of local public goods provision (e.g., Bramoullé and Kranton, 2007). For instance, when individuals experiment with a new technology or information, they exert positive externalities on each other but efforts are substitutes (the more their neighbor experiments, the less an individual needs to experiment).

   The case of a negative ϕ^s, that is, efforts are strategic complements and neighbors exert negative externalities on each other, is less standard since the canonical model (Ballester et al., 2006) assumes both complementarities and positive spillovers. However, in many real-world situations, we observe these effects. Consider, for example, the crime committed individually by criminals. Criminal efforts are clearly strategic complements since the more a criminal makes an effort, the higher is the utility of a connected criminal in making effort. Indeed, delinquents learn from other criminals who are connected to them on how to commit crime in a more efficient way by sharing the know-how about the technology of crime. Moreover, each criminal exerts a negative spillover on other criminals because they compete for the same resources. For instance, when someone commits a burglary or robs people, it has a negative externality on other criminals in the same neighborhood since the common resources have shrunk for the remaining criminals. In fact, Calvó-Armengol and Zenou (2004) and Ballester et al. (2006) also considered this effect but assumed that it is global; however, in this study, we assume it is local and through the network.

Another interpretation of our assumptions is conspicuous consumption/social status (Ghiglino and Goyal, 2010). By buying a "better" or more prestigious car than other individuals in the network, an agent creates a negative externality on these people. However, it also induces them to want to buy a "better" car (strategic complementarity).

Furthermore, it is possible that $\phi^{s'} > 0$ on a layer s' but $\phi^{s''} < 0$ on a different layer s''.

4. Observe that x_i^s and v_i^s are strategic substitutes since

$$\frac{\partial^2 U_i}{\partial x_i^s \partial v_i^s} = -\frac{\alpha^s}{(v_i^s + x_i^s + \phi^s \sum_{j \in \mathcal{N}} g_{ij}^s x_j^s)^2} < 0. \tag{4}$$

In this respect, v_i^s can be interpreted as an endowment, so that when v_i^s increases, the marginal benefit of increasing own effort v_i^s decreases.

5. In the baseline model, for simplicity, we do not allow spillovers across different layers, i.e., whenever $t \neq s$,

$$\frac{\partial^2 U_i}{\partial x_i^s \partial x_j^t} = 0, \forall i, j \in \mathcal{N}.$$

6. Regarding the Cobb-Douglas utility function (1), we can easily generalize it to a CES utility function given by

$$U_i(\mathbf{x}_i, \mathbf{x}_{-i}) = \left[\sum_{s \in \mathcal{S}} \alpha^s \left(v_i^s + x_i^s + \phi^s \sum_{j \in \mathcal{N}} g_{ij}^s x_j^s \right)^\rho \right]^{1/\rho},$$

where $0 < \rho < 1$. Our main results will remain the same under this CES specification.

7. For each layer s, we assume that the set of agents $\mathcal{N}$ is the same. This is without loss of generality since we allow for heterogeneous social networks. In addition, we assume that α^s is layer-specific and not individual-specific; that is, each agent has the same preference weight for the same layer. This also implies that agents care about all layers.

2.2 Interpretations/applications

The model allows for different interpretations, depending on the economic contexts. Here we list two main applications:

(i) **Management of multiple social relationships:** Agents allocate time/attention across different layers of social connections, for instance, their family network, professional network, and religious network. The utility takes the following form:

$$U_i(\mathbf{x}_i, \mathbf{x}_{-i}) = \sum_{s \in \mathcal{S}} \alpha^s \ln \left(v_i^s + x_i^s + \phi^s \sum_{j \in \mathcal{N}} g_{ij}^s x_j^s \right). \tag{5}$$

Each individual i solves the following problem:

$$\max_{\mathbf{x}_i \in \mathbf{R}_+^s} \sum_{s \in \mathcal{S}} \alpha^s \ln \left(v_i^s + x_i^s + \phi^s \sum_{j \in \mathcal{N}} g_{ij}^s x_j^s \right) \quad \text{s.t.} \sum_{s \in \mathcal{S}} x_i^s = T_i, \tag{6}$$

where T_i is the maximum time each agent i has, so $\sum_{s \in \mathcal{S}} x_i^s = T_i$ is the time constraint of agent i.

(ii) **Multiple public goods:** There are three goods: the first good is a local public good x_i^1 with network $\mathbf{G}$ and spillover parameter $\phi^1 > 0$, the second good is a pure private good x_i^2 (the corresponding network is thus empty), and the third good is a global pure public good x_i^3 (the corresponding network structure is thus complete with $\phi^3 = 1$). The utility function of each agent i is then given by

$$U_i(\mathbf{x}_i, \mathbf{x}_{-i}) = \alpha^1 \ln \left(v_i^1 + x_i^1 + \phi^1 \sum_{j \in \mathcal{N}} g_{ij} x_j^1 \right) + \alpha^2 \ln x_i^2 + \alpha^3 \ln \sum_{k \in \mathcal{N}} x_k^3, \tag{7}$$

where $\sum_{k \in \mathcal{N}} x_k^3$ is the total provision of the global public good. The income of agent i is just T_i. Each individual i solves the following problem:[7]

$$\max_{x_i^1, x_i^2, x_i^3 \geq 0} \left\{ \alpha^1 \ln \left(v_i^1 + x_i^1 + \phi^1 \sum_{j \in \mathcal{N}} g_{ij} x_j^1 \right) + \alpha^2 \ln x_i^2 + \alpha^3 \ln \sum_{k \in \mathcal{N}} x_k^3 \right\} \text{ s.t. } x_i^1 + x_i^2 + x_i^3 = T_i. \tag{8}$$

Our model nests these particular models as special cases. We will use these applications to illustrate our main results.

[7]Note that, for simplicity, we have normalized the prices of the three goods to 1.

3 Equilibrium characterizations

3.1 A (best-reply) potential game approach

For each layer s, define $\mathbf{x}^s = (x_1^s, \cdots, x_n^s)^T \in \mathbf{R}_n$ as the action profile on layer s and $\mathbf{v}^s = (v_1^s, \cdots, v_n^s)^T \in \mathbf{R}_n$ as the endowment vector. Define a function θ on $K = \prod_i K_i$, where

$$-\theta(\mathbf{X}) = \sum_{s\in\mathcal{S}} \frac{1}{2\alpha^s} \left\{ \sum_{i\in\mathcal{N}} (x_i^s)(2v_i^s + x_i^s + \phi^s \sum_{j\in\mathcal{N}} g_{ij}^s x_j^s) \right\} = \sum_{s\in\mathcal{S}} (\frac{1}{2\alpha^s})(\mathbf{x}^s)'(2\mathbf{v}^s + (\mathbf{I}_n + \phi^s \mathbf{G}^s)\mathbf{x}^s). \quad (9)$$

From this construction, we note that this function θ of $\mathbf{X}$ depends only on the model primitives: $\mathbf{v}^s, \alpha^s, \mathbf{G}^s, \phi^s, s \in \mathcal{S}$.

Theorem 1. *θ is a best-reply potential of the multiplexity network game. Specifically,*

$$\arg\max_{\mathbf{x}_i\in K_i} U_i(\mathbf{x}_i, \mathbf{x}_{-i}) = \arg\max_{\mathbf{x}_i\in K_i} \theta(\mathbf{x}_i, \mathbf{x}_{-i}), \quad \textit{for any } \mathbf{x}_{-i} \in K_{-i}. \quad (10)$$

Thus, any global maximizer $\mathbf{X}^*$ of $\theta(\mathbf{X})$ in K is a pure strategy Nash equilibrium.

In Theorem 1, we establish that the multiplexity network game has a best-reply potential θ. The main intuition behind Theorem 1 is to construct *suitable* transformations of the underlying equilibrium conditions of the orginal game such that the new conditions exactly match the optimality conditions of maximizing θ on K. In the proof of Theorem 1, we first establish a lemma (Lemma 1), which shows that the (unique) solution of the optimization problem $\max_{\mathbf{x}\in\Delta} \sum_{s\in\mathcal{S}} \alpha^s \ln(v^s + x^s)$[8] is exactly the same as that of the optimization problem $\max_{\mathbf{x}\in\Delta} -\sum_{s\in\mathcal{S}} \frac{1}{2\alpha^s}(v^s + x^s)^2$. This implies that, fixing any $\mathbf{x}_{-i}$, maximizing i's utility function (1) under the constraint $\sum_{s\in\mathcal{S}} x_i^s = T_i$ yields the same solution as maximizing the function θ defined in (9) over $\mathbf{x}_i$ for the same constraint. The construction of θ guarantees that we can apply Lemma 1 for every agent i, which proves Theorem 1. Note that the game *does not* have an exact potential in the sense of Monderer and Shapley (1996).

Theorem 1 is useful for both establishing equilibrium existence and identifying conditions for uniqueness. Clearly, θ is continuous and the domain K is compact. A global maximizer $\mathbf{X}^*$ of $\theta(\cdot)$ in K must exist, which, by Theorem 1, must be an equilibrium. To obtain uniqueness, we need to impose some mild technical conditions.

[8]where Δ is the set of nonnegative vectors $\mathbf{x} = (x^1, \cdots, x^s) \geq \mathbf{0}$ satisfying $\sum_s x^s = T > 0$.

Assumption 1. *The quadratic form*

$$Q(\mathbf{z}^1, \cdots, \mathbf{z}^s) = \sum_{s\in\mathcal{S}} \left(\frac{1}{2\alpha^s}\right) (\mathbf{z}^s)'(\mathbf{I}_n + \phi^s \mathbf{G}^s)\mathbf{z}^s \tag{11}$$

is positive definite subject to the linear constrains $\sum_{s\in\mathcal{S}} \mathbf{z}^s = \mathbf{0}_n$ (here $\mathbf{z}^s \in \mathbf{R}^n, \forall s \in \mathcal{S}$).

Proposition 1.

(i) Under Assumption 1, the equilibrium is unique.

(ii) Suppose $1 + \lambda_{\min}(\phi^s \mathbf{G}^s) > 0, \forall s \in \mathcal{S}$. Then, Assumption 1 holds.

Proposition 1(i) is intuitive. Under the stated assumption, θ is strictly concave on K as Q defined in (11) is the negative of the quadratic term of θ, and the linear constraints in Assumption 1 are due to the binding budget equations of agents. The condition in Proposition 1(ii) for each fixed layer s is standard in the monolayer network literature. For instance, when $\phi^s < 0$, it reduces to $\lambda_{\max}(\mathbf{G}^s) < \frac{1}{\phi^s}$ (Ballester et al., 2006), while when $\phi^s > 0$, it reduces to $\lambda_{\min}(\mathbf{G}^s) > \frac{-1}{\phi^s}$ (Bramoullé and Kranton, 2007; Bramoullé et al., 2014). Furthermore, the condition in item (ii) is sufficient for Assumption 1 but often not necessary.[9] We maintain this assumption throughout the paper.

3.2 Within-layer influence and cross-layer interaction

Agents' incentives are quite complex in our multiplex network game as each agent, while allocating efforts within their budget, needs to take into account externalities imposed by other agents on every layer. To better understand these incentives, we perform a decomposition into (i) the within-layer influence and (ii) the cross-layer interaction.

The *within-layer* influence is a well-studied topic in the monolayer network literature. For each $s \in S$, we define the following inverse Leontief matrix:

$$\mathbf{M}^s = [\mathbf{I}_n + \phi^s \mathbf{G}^s]^{-1}. \tag{12}$$

[9]For instance, when there are two layers ($|\mathcal{S}| = 2$), Assumption 1 reduces to $(\frac{1}{\alpha^1})(\mathbf{I}_n + \phi^1\mathbf{G}^1) + (\frac{1}{\alpha^2})(\mathbf{I}_n + \phi^2\mathbf{G}^2)$ being positive definite. Or, equivalently, $\lambda_{\min}(\frac{\phi^1}{\alpha^1}\mathbf{G}^1 + \frac{\phi^2}{\alpha^2}\mathbf{G}^2) > \frac{1}{1/\alpha^1 + 1/\alpha^2}$. This requirement is weaker than that imposed in Proposition 1(ii).

This matrix (or its variant) appears in several monolayer network models (e.g., Ballester et al., 2006; Bramoullé et al., 2014), as it counts the discounted number of walks between nodes in layer s. Formally, when $1 + \lambda_{\min}(\phi^s \mathbf{G}^s) > 0$ (see Proposition 1(ii)), we obtain the following infinite series:

$$\mathbf{M}^s = \sum_{k \geq 0} (-\phi^s \mathbf{G}^s)^k.$$

To capture *cross-layer* linkages, we introduce a new *multiplexity influence measure*, defined as follows:

Definition 1. *Define $\boldsymbol{\mu}^* = (\mu_1^*, \cdots, \mu_n^*)'$, the system-level centrality, with*

$$\boldsymbol{\mu}^* = \left(\sum_{s \in \mathcal{S}} \alpha^s \mathbf{M}^s \right)^{-1} \left(\mathbf{T} + \sum_{s \in \mathcal{S}} \mathbf{M}^s \mathbf{v}^s \right), \text{ where } \mathbf{T} = (T_1, \cdots, T_n)'. \tag{13}$$

For each agent i, μ_i depends on i's influence on layer s as measured by the entries in $\mathbf{M}^s$, with some aggregation across all layers by preference weights α^s taking into account i's constraint T_i and the endowments $\mathbf{v}^s$. Note that $\boldsymbol{\mu}^*$ depends only on the model primitives.

The following proposition illustrates how the Leontief matrix $\mathbf{M}^s$ of within-layer influence, endowments $\mathbf{v}^s$, and the system-level centrality $\boldsymbol{\mu}^*$ of between-layer linkages fully shape the equilibrium:

Proposition 2. *At an interior equilibrium $\mathbf{X}^*$, the following holds:*[10]

(i) The action profile on layer $s \in \mathcal{S}$ is equal to

$$\mathbf{x}^{s^*} = \mathbf{M}^s (\alpha^s \boldsymbol{\mu}^* - \mathbf{v}^s), \tag{14}$$

where $\boldsymbol{\mu}^*$ is defined by (13).

(ii) The equilibrium payoff of agent i satisfies

$$U_i^* = \left(\sum_{s \in \mathcal{S}} \alpha^s \right) \cdot \ln(\mu_i^*) + \sum_{s \in \mathcal{S}} \alpha^s \ln(\alpha^s). \tag{15}$$

Proposition 2 is instrumental for the subsequent comparative statics results as it simplifies the analysis of the equilibrium. On each layer s, $\mathbf{x}^{s^*}$ is proportional to the Katz-Bonacich centrality with weight vector $\alpha^s \boldsymbol{\mu}^* - \mathbf{v}^s$. As shown in the proof of

[10] See Appendix B for a discussion on the interiority of the equilibrium efforts.

Proposition 2, the weight for each agent i is $\mu_i^* = 1/\lambda_i^*$, where λ_i^* is the (Lagrangian) multiplier of i's budget constraint. Note that, on each layer s, the following identity holds in an interior equilibrium:

$$\underbrace{\mathbf{v}^s + \mathbf{x}^{s*} + \phi^s \mathbf{G}^s \mathbf{x}^{s*}}_{:=\mathbf{q}^{s*}} = \alpha^s \boldsymbol{\mu}^*.$$

That is, the "effective efforts" $\mathbf{q}^{s*}$, which combine the endowments $\mathbf{v}^s$, own efforts $\mathbf{x}^{s^*}$, and the spillover efforts from neighbors $\phi^s \mathbf{G}^s \mathbf{x}^{s*}$, must be proportional to the multiplexity influence $\boldsymbol{\mu}^*$. This is intuitive given that an agent's marginal utility of efforts for any two layers s and t must be equal, that is, $\frac{\alpha^s}{q_i^{s*}} = \frac{\alpha^t}{q_i^{t*}}$. In vector form, $\frac{\mathbf{q}^{s*}}{\alpha^s} = \frac{\mathbf{q}^{t*}}{\alpha^t}$. Thus, the equilibrium effective efforts are higher for layers with a larger preference weight for the layer.

The multiplier λ_i^*s, or μ_i^*'s, is *endogenously* determined in equilibrium by the agent's resource constraints and the interactions within and across layers. We can then pin down μ_i^* using i's budget constraint, which leads to (13).

As a by-product, Proposition 2(ii) reveals that $\boldsymbol{\mu}^*$ is closely related to the agents' equilibrium payoff:

Corollary 1. *Under the same assumption as in Proposition 2, $U_i^* \geq U_j^*$ if and only if $\mu_i^* \geq \mu_j^*$.*

That is, the player with the highest μ_i index obtains the highest payoff. Agents' efforts are multidimensional; yet, μ_i^* provides a summary statistics of equilibrium payoffs. In particular, player i with the highest equilibrium payoff is the one with the highest μ_i^*.

The agents in our model are heterogeneous in terms of both their endowments of resources and their network positions in different layers. Below, we will use the results in Proposition 2 to illustrate the main mechanisms behind shock propagation in multilayer networks. To do so, we first need to pin down the effects on μ^* since, once μ^* is determined, the comparative statics results on each layer s follow from the standard analysis with a single layer.

3.3 Two applications

3.3.1 Management of multiple social relationships

Consider application (i) in Section 2.2. If we denote by λ_i the Lagrangian multiplier of the time constraint, by solving (6), we easily obtain

$$v_i^s + x_i^s + \phi^s \sum_{j\in\mathcal{N}} g_{ij}^s x_j^s = \alpha^s \mu_i, \tag{16}$$

where $\mu_i := 1/\lambda_i$. By writing this equation in vector-matrix form, we obtain (14). By plugging this effort into utility (5), we obtain (15). Then, given the fact that $\sum_{s\in\mathcal{S}} \mathbf{x}^s = \mathbf{T}$, we get $\mathbf{T} = \sum_{s\in\mathcal{S}} \mathbf{M}^s(\alpha^s\boldsymbol{\mu}^* - \mathbf{v}^s)$, which leads to equation (13).

3.3.2 Multiple public goods

Consider now application (ii) in Section 2.2.[11] Solving (8) for each individual i leads to the following first-order conditions in an interior solution:[12]

$$v_i^1 + x_i^1 + \phi^1 \sum_{j\in\mathcal{N}} g_{ij} x_j^1 = \alpha^1 \mu_i, \tag{17}$$

$$x_i^2 = \alpha^2 \mu_i, \tag{18}$$

$$\sum_{k\in\mathcal{N}} x_k^3 = \alpha^3 \mu_i. \tag{19}$$

where $\mu_i := 1/\lambda_i$. Note that layer $s = 1$ corresponds to the local public good, layer $s = 2$ to the pure private good, and layer $s = 3$ to the global pure public good.

Since $\sum_{k\in\mathcal{N}} x_k^3 = \alpha^3\mu_i$, it has to be the case that $\mu_1^* = ... = \mu_n^* = \mu^*$, which further implies that $x_1^2 = ... = x_n^2 = x^2$. For the local public good, we have

$$\mathbf{x}^1 = \left[\mathbf{I}_n + \phi^1\mathbf{G}\right]^{-1}\left(\alpha^1\mu^*\mathbf{1} - \mathbf{v}^1\right) = \mathbf{M}^1\left(\alpha^1\mu^*\mathbf{1} - \mathbf{v}^1\right).$$

By plugging these three different good consumptions into utility (5), we easily obtain

$$\ln U_i^* = \ln\mu^* \sum_{s=1}^{s=3}\alpha^s + \sum_{s=1}^{s=3}\alpha^s \ln\alpha^s, \tag{20}$$

[11] In this application, Assumption 1 holds if and only if $1 + \phi^1\lambda_{min}(\mathbf{G}) > 0$.

[12] Proposition 2 is not applicable here as when $\phi^3 = 1$, the matrix $\mathbf{I}_n + \phi^3\mathbf{G}^3$ is composed of 1 everywhere, which is singular (rank = 1) and thus not invertible. Therefore, $\mathbf{M}^3 = (\mathbf{I}_n + \phi^3\mathbf{G}^3)^{-1}$ is not well-defined. This is why here we solve the model in a different way. Furthermore, in the simulations, the unique equilibrium is indeed interior.

which implies that the equilibrium utility is the same for all players, i.e., $U_1^* = ... = U_n^* = U^*$ (Corollary 1). Finally, by using the budget constraint $x_i^1 + x_i^2 + x_i^3 = T_i$, we find that

$$\mu^* = \frac{\sum_{i\in\mathcal{N}} T_i + \mathbf{1}^T\mathbf{M}^\mathbf{1}\mathbf{v}^\mathbf{1}}{\alpha^1\mathbf{1}^T\mathbf{M}^1\mathbf{1} + n\alpha^2 + \alpha^3}. \tag{21}$$

In summary, in this multiple public-goods application, we find that, in equilibrium, all agents have the same utility, consume the same amount of private and global public goods, but differ in their contribution of local public good.

Remark 1. *It is the presence of the global public good that equalizes the consumptions of agents. Therefore, the result that all agents obtain the same utility and consume the same amount of private and global public goods would still be true even if we had k local public goods with k different networks instead of one local public good and one network.*

Remark 2. *The standard models of public good are special cases in this study. Indeed, when $\alpha^3 = 0$, we obtain both the models of Bramoullé and Kranton (2007) and Allouch (2015)*[13] *and when $\alpha^1 = 0$, we obtain the model of Bergstrom et al. (1986). In the case of $\alpha^3 = 0$, we do not obtain the result that all agents obtain the same utility and consume the same amount of private and global public goods. It is the network multiplexity of our model that introduces a new network of global public good that yields this result.*

Consider a policy in which the planner who maximizes aggregate welfare (the sum of the utilities of all agents) has to decide to which player i they want to give an income transfer. Let t_i denote the income transfer made to consumer i, which may be either a tax ($t_i < 0$) or a subsidy ($t_i \geq 0$), and let the transfer vector be denoted by $\mathbf{t} = (t_1, ..., t_n)'$, which lists all income transfers made to consumers. Every transfer is budget balanced, that is, $\sum_{i\in\mathcal{N}} t_i = 0$. The new budget for each individual i is now $T_i + t_i$.

Definition 2. *A transfer $\mathbf{t}$ is neutral if it leaves unchanged the consumption of the private and public goods for each consumer i.*

Remark 3. *As can be seen from (20) and (21), the utility of each agent only depends on $\sum_{i\in\mathcal{N}} T_i$. Thus, the planner is indifferent to which individual to give the transfer;*

[13] Their utility function is more general but the results will be similar.

any budget-balanced transfer $\mathbf{t}$ *is neutral.*[14]

Remark 4. *If we assume that* $0 < \phi^3 < 1$ *instead of* $\phi^3 = 1$*, as we do here, then these results do not hold anymore. If* $\phi^3 < 1$*, we obtain qualitatively the same results as in the application of the management of multiple social relationships described in Section 3.3.1.*

3.4 Same network between layers

Consider the special case in which the network is the same between layers, i.e., $\mathbf{G}^s = \mathbf{G}$ for all $s \in \mathcal{S}$. Observe that we can have any network structure and ϕ can be different between layers, i.e., $\phi^s \neq \phi^{s'}$, for $s \neq s'$.

In this case, we can extend the principle component decomposition technique of Galeotti et al. (2020) and Chen et al. (2022) to derive simpler equilibrium characterizations in our multilayer setting.[15] By the spectral theorem of symmetric matrices, we have

$$\mathbf{G} = \mathbf{Z}\, diag(\gamma_1, \cdots, \gamma_n)\mathbf{Z}^{-1},$$

with $\mathbf{Z} = (\mathbf{z}_1, \cdots, \mathbf{z}_n)$ and where each column $\mathbf{z}_k$ is the normalized eigenvector of $\mathbf{G}$ for the eigenvalue γ_k, i.e., $\mathbf{G}\mathbf{z}_k = \gamma_k \mathbf{z}_k, k = 1, \cdots, n$.[16]

Define $\widehat{\mathbf{T}} = \mathbf{Z}^{-1}\mathbf{T}$, $\widehat{\boldsymbol{\mu}}^* = \mathbf{Z}^{-1}\boldsymbol{\mu}^*$, $\widehat{\mathbf{x}}^{s*} = \mathbf{Z}^{-1}\mathbf{x}^{s^*}$, and $\widehat{\mathbf{v}}^s = \mathbf{Z}^{-1}\mathbf{v}^s$, for each $s \in \mathcal{S}$. We have the following result:

Proposition 3. *In the eigenspace of* $\mathbf{G}$*, the equilibrium allocation of agent* i *is given by:*

$$\widehat{x}_i^{s*} = \frac{1}{1 + \phi^s \gamma_i} \left(\alpha^s \widehat{\mu}_i^* - \widehat{v}_i^s \right), \tag{22}$$

where

$$\widehat{\mu}_i^* = \left(\sum_{s \in \mathcal{S}} \frac{\alpha^s}{1 + \phi^s \gamma_i} \right)^{-1} \left(\widehat{T}_i + \sum_{s \in \mathcal{S}} \frac{1}{1 + \phi^s \gamma_i} \widehat{v}^s \right). \tag{23}$$

First, observe that μ_i^*, defined by (23), only depends on agent i's information through $\widehat{v}_i^s$ and $\widehat{T}_i$, but not on the information of agent j through $\widehat{v}_j^s$ and $\widehat{T}_j$ for $j \neq i$. In

[14] As pointed out by Bergstrom et al. (1986), the transfer should be local, i.e., not too big to affect the set of contributors in equilibrium.

[15] We can apply a similar method for the case with heterogeneous networks $\mathbf{G}^s, s \in \mathcal{S}$ as long as these set of network matrices commute, i.e., $\mathbf{G}^s\mathbf{G}^t = \mathbf{G}^t\mathbf{G}^s, \forall t, s$.

[16] Note that $\mathbf{Z}$ is orthogonal by construction.

other words, in the eigenspace of $\mathbf{G}$, different agents act as if they were independent of each other.

Second, the expression of x_i^{s*} in (22)[17] is much simpler than the one obtained in the general case (see equation (14) in Proposition 2) because instead of having $\mathbf{M}^s = [\mathbf{I}_n + \phi^s \mathbf{G}^s]^{-1}$, a complex matrix of within-layer influence, which counts the discounted number of walks between nodes in layer s, we have instead $(1 + \phi^s \gamma_i)^{-1}$, which only depends on ϕ^s, the intensity of spillover effects, and γ_i, the eigenvalue associated with the eigenvector of the ith column of $\mathbf{Z}$. Clearly, using this simpler expression, we could obtain sharper results, in particular, in Section 4.3 below on targeting.[18]

Remark 5. *Consider the more restrictive case in which (i) for each $s \in \mathcal{S}$, $\phi^s \mathbf{G}^s = \phi \mathbf{G}$,[19] and $\mathbf{v}^s = 0$ for any $s \in \mathcal{S}$. Then, there exists an equilibrium in which*

$$x_i^{s*} = \frac{\alpha^s}{\sum_{k \in \mathcal{S}} \alpha^k} T_i, \ \forall i \in N, s \in \mathcal{S}.$$

In this setting, we have $\mathbf{M}^s = \mathbf{M}^t$ for any s and t. Due to the symmetry of network influence across layers, the agents allocate efforts purely based on preference weights α. This example shows that network influence, in itself, does not always affect effort allocation between layers.

3.5 Regular networks

To obtain even more explicit results, let us focus on regular networks.[20] We impose the following conditions:

- For each $s \in \mathcal{S}$, $\mathbf{G}^s$ is a regular network with degree d^s, i.e., $\mathbf{G}^s \mathbf{1} = d^s \mathbf{1}$;
- $T_i = T, \forall i \in \mathcal{N}$;
- $\mathbf{v}^s = 0$ for any $s \in \mathcal{S}$.

[17] Using $\widehat{\boldsymbol{\mu}}^* = \mathbf{Z}^{-1}\boldsymbol{\mu}^*$, $\widehat{\mathbf{x}}^{s*} = \mathbf{Z}^{-1}\mathbf{x}^{s^*}$, and $\widehat{\mathbf{v}}^s = \mathbf{Z}^{-1}\mathbf{v}^s$, it is straightforward to obtain x_i^{s*} from $\widehat{x}_i^{s*}$ defined in (22).

[18] We can have simpler targeting indices $\boldsymbol{\eta}$ and $\boldsymbol{\tau}^s$ defined in Proposition 6. The details are available upon request.

[19] A special case is $\phi^s = \phi, \mathbf{G}^s = \mathbf{G}$ for any s.

[20] This assumption does not imply that the regularity (i.e., the degree of each agent) is the same between layers.

These assumptions lead to some symmetry of the players, but still allow for layer heterogeneity in terms of $\{d^s, \phi^s, \alpha^s\}$. See Figure 1 for an illustration with four players where layer 1 is a complete network, layer 2 is composed of two dyads, and layer 3 is the empty network; all networks are regular.

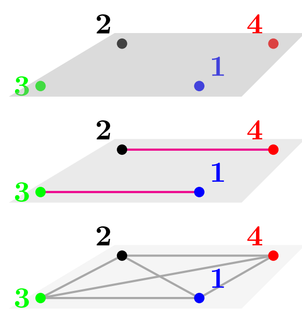

Figure 1: An example with a regular network in each layer.

In the general case of regular networks, it is easy to show that there exists an equilibrium in which $x_i^s = x^{s*}, \forall i \in N, s \in \mathcal{S}$, where

$$x^{*s} = \frac{\alpha^s/[1+\phi^s d^s]}{\sum_{s'}\{\alpha^{s'}/[1+\phi^{s'} d^{s'}]\}}T. \tag{24}$$

In particular, in equilibrium, for two layers s and t,

$$\frac{x^{s*}}{x^{t*}} = \frac{\alpha^s/[1+\phi^s d^s]}{\alpha^t/[1+\phi^t d^t]}. \tag{25}$$

In other words, the common equilibrium action in a layer $s \in \mathcal{S}$ is proportional to the product of the utility weight α^s and the social multiplier $1/[1+\phi^s d^s]$.

Proposition 4. *Consider regular networks. Then,*

1. *x^{s*} increases with α^s and decreases with α^t, for $t \neq s$;*

2. *x^{s*} decreases (increases) with d^s when $\phi^s > (<)0$.*

3. *x^{s*} increases (decreases) with d^t when $\phi^t > (<)0$, for $t \neq s$.*

Quite naturally, the effort in each layer is increasing in the utility weight in the same layer and decreasing with the weight in any other layer. In addition, if there are strategic complementarities in efforts, the effort in a given layer increases with the number of links in this layer and decreases with the number of links in the other layers. The result is reversed for strategic substitutes.

3.6 Inefficiency of equilibrium allocations

Define

$$W(\mathbf{X}) := \sum_{i\in\mathcal{N}} U_i(\mathbf{X})$$

as the aggregate payoff in equilibrium (total welfare). That is,

$$W(\mathbf{X}) = \sum_{i\in\mathcal{N}}\sum_{s\in\mathcal{S}} \alpha^s \ln\left(v_i^s + x_i^s + \phi^s \sum_{j\in\mathcal{N}} g_{ij}^s x_j^s\right). \tag{26}$$

An allocation $\hat{\mathbf{X}} \in K = \prod_{i\in\mathcal{N}} K_i$ is called efficient if $W(\hat{\mathbf{X}}) \geq W(\mathbf{X})$ for any $\mathbf{X} \in \prod_{i\in\mathcal{N}} K_i$. The efficient allocation exists and it must be unique given the strict concavity of $W(\cdot)$ and the convexity and compactness of the choice set K. Determining the first-order conditions with respect to $x_i^s, i \in \mathcal{N}, s \in \mathcal{S}$ leads to the next proposition:

Proposition 5. *An interior allocation $\hat{\mathbf{X}}$ is efficient if and only if it satisfies the following equality:*

$$\frac{\alpha^s}{(v_i^s + x_i^s + \phi^s \sum_{j\in\mathcal{N}} g_{ij}^s x_j^s)} + \sum_{j\neq i} \frac{\alpha^s \phi^s g_{ji}^s}{(v_j^s + x_j^s + \phi^s \sum_{k\in\mathcal{N}} g_{jk}^s x_k^s)} = \hat{\lambda}_i, \quad \forall i \in \mathcal{N}, s \in \mathcal{S} \tag{27}$$

where $\hat{\lambda}_i$ is the multiplier of agent i's budget constraint.

Comparing the conditions for an equilibrium allocation and those for the efficient allocation gives the discrepancy between these two allocations as each agent i in equilibrium does not take into account the effects of x_i^s on other agents' payoff. Basically, the first-order condition for each agent i at the Nash equilibrium corresponds to (27) without the second term on the left-hand side of this equation (see (16) with $\mu_i := 1/\hat{\lambda}_i$). When ϕ^s is positive (negative), agent i underestimates (overestimates) the marginal welfare effects of x_i^s. Since the aggregate budget is fixed for each player i, the discrepancy boils down to the relative allocations across different layers. Given the wedge between different private returns of effort and the social return, $\mathbf{X}^*$ is unlikely to be efficient in general.

To obtain some intuition, we consider again regular networks (Section 3.5).

Example 1. *Consider regular networks with $v_i^s = 0$. Then, the unique efficient allocation satisfies $\hat{x}_i^s = \hat{x}^s, \forall i \in \mathcal{N}$, where*

$$\hat{x}^s = \frac{\alpha^s}{\sum_{t\in\mathcal{S}} \alpha^t} T.$$

In particular, for two layers $s \neq t$,

$$\frac{\hat{x}^s}{\hat{x}^t} = \frac{\alpha^s}{\alpha^t}. \tag{28}$$

The equilibrium allocation $\mathbf{X}^*$ (25) *satisfies* $x_i^s = x^{*s}, \forall i \in N, s \in \mathcal{S}$, *where*

$$\frac{x^{*s}}{x^{*t}} = \frac{\alpha^s/[1+\phi^s d^s]}{\alpha^t/[1+\phi^t d^t]}. \tag{29}$$

By comparing (28) *and* (29), *we observe a discrepancy between* x^{s*} *and* $\hat{x}^s$. *Interestingly, the efficient allocation* $\hat{\mathbf{X}}^s$ *does not depend on the degrees* d^s *and the network effect parameters* ϕ^s *while the equilibrium allocation* $\mathbf{X}^*$ *depends on these two parameters.*

4 Policy implications and shock propagations

4.1 General results

Consider a planner that chooses the level of subsidies v_i^s and income T_i for each agent i in each layer s to maximize aggregate welfare (26). In particular, we can study how $\mathbf{v}^s, s \in \mathcal{S}$ and $\mathbf{T}$ affect the equilibrium allocation, payoffs, and aggregate welfare. Denote $\mathbf{U}^* = (U_1, \cdots, U_n)^T$ as the equilibrium payoff vector.

Theorem 2. *At an interior equilibrium* $\mathbf{X}^*$, *the following relationships hold*

(i) Effects of $\mathbf{v}^s$:

$$\frac{\partial \boldsymbol{\mu}^*}{\partial \mathbf{v}^s} = \left(\sum_{t\in\mathcal{S}} \alpha^t \mathbf{M}^t\right)^{-1} \mathbf{M}^s. \tag{30}$$

$$\frac{\partial \mathbf{x}^{s*}}{\partial \mathbf{v}^s} = -\mathbf{M}^s + \alpha^s \mathbf{M}^s \left(\sum_{t\in\mathcal{S}} \alpha^t \mathbf{M}^t\right)^{-1} \mathbf{M}^s. \tag{31}$$

$$\frac{\partial \mathbf{x}^{s'*}}{\partial \mathbf{v}^s} = \alpha^{s'} \mathbf{M}^{s'} \left(\sum_{t\in\mathcal{S}} \alpha^t \mathbf{M}^t\right)^{-1} \mathbf{M}^s, \textit{ for } s' \neq s. \tag{32}$$

$$\frac{\partial \mathbf{U}^*}{\partial \mathbf{v}^s} = \left(\sum_{s\in\mathcal{S}} \alpha^s\right) diag(1/\mu_1^*, \cdots, 1/\mu_n^*) \left(\sum_{t\in\mathcal{S}} \alpha^t \mathbf{M}^t\right)^{-1} \mathbf{M}^s. \tag{33}$$

$$\frac{\partial W^*}{\partial \mathbf{v}^s} = \left(\sum_{s\in\mathcal{S}} \alpha^s\right) (1/\mu_1^*, \cdots, 1/\mu_n^*) \left(\sum_{t\in\mathcal{S}} \alpha^t \mathbf{M}^t\right)^{-1} \mathbf{M}^s. \tag{34}$$

(ii) Effects of $\mathbf{T}$*:*

$$\frac{\partial \boldsymbol{\mu}^*}{\partial \mathbf{T}} = \left(\sum_{t\in\mathcal{S}} \alpha^t \mathbf{M}^t\right)^{-1}. \tag{35}$$

$$\frac{\partial \mathbf{x}^{s*}}{\partial \mathbf{T}} = \alpha^s \mathbf{M}^s \left(\sum_{t\in\mathcal{S}} \alpha^t \mathbf{M}^t\right)^{-1}. \tag{36}$$

$$\frac{\partial \mathbf{U}^*}{\partial \mathbf{T}} = \left(\sum_{s\in\mathcal{S}} \alpha^s\right) diag(1/\mu_1^*, \cdots, 1/\mu_n^*) \left(\sum_{t\in\mathcal{S}} \alpha^t \mathbf{M}^t\right)^{-1}. \tag{37}$$

$$\frac{\partial W^*}{\partial \mathbf{T}} = \left(\sum_{s\in\mathcal{S}} \alpha^s\right) (1/\mu_1^*, \cdots, 1/\mu_n^*) \left(\sum_{t\in\mathcal{S}} \alpha^t \mathbf{M}^t\right)^{-1}. \tag{38}$$

Theorem 2 is a direct consequence of the equilibrium characterization results in Proposition 2. The intuition of the results is relatively simple. First, consider the effect of a change in $\mathbf{v}^s$ on $\mathbf{x}^{s*}$, the individual efforts in the same layer s. By differentiating (14), we obtain

$$\frac{\partial \mathbf{x}^{s*}}{\partial \mathbf{v}^s} = -\mathbf{M}^s + \alpha^s \mathbf{M}^s \frac{\partial \boldsymbol{\mu}^*}{\partial \mathbf{v}^s}.$$

That is, when the planner increases $\mathbf{v}^s$, there is a *direct negative effect* on $\mathbf{x}^s$ captured by the matrix $\mathbf{M}^s$, defined in (12), because $\mathbf{v}^s$ and $\mathbf{x}^s$ are strategic substitutes. The new part is the second term that captures the *indirect effects* from the other layers through the impact of $\mathbf{v}^s$ on $\boldsymbol{\mu}^*$. When someone decreases their effort in one layer, they need to adjust their effort in the other layers depending on the attractiveness of each layer. This is captured by $\frac{\partial \boldsymbol{\mu}^*}{\partial \mathbf{v}^s}$, which is obtained by differentiating (13) in Definition 1. Second, consider the effect of a change in $\mathbf{v}^s$ on another layer $s' \neq s$. In this case, only the indirect effect through $\boldsymbol{\mu}^*$ matters. Finally, since there is a one-to-one relationship between equilibrium payoff and μ_i (Corollary 1), the impacts of $\mathbf{v}^s$ on utility $\mathbf{U}^*$ and welfare W^* are similar.

Consider now the effect of a change in $\mathbf{T}$. First, the impact of $\mathbf{T}$ on $\mathbf{x}^{s*}$ is only through $\boldsymbol{\mu}^*$. When $\mathbf{T}$ increases, agents have more resources and thus can allocate their efforts differently across layers, which affects their shadow prices of the budget. Indeed, differentiating (13) with respect to $\mathbf{T}$ yields $\frac{\partial \boldsymbol{\mu}^*}{\partial \mathbf{T}} = \left(\sum_{t\in\mathcal{S}} \alpha^t \mathbf{M}^t\right)^{-1}$, which, by (14), leads to $\frac{\partial \mathbf{x}^{s*}}{\partial \mathbf{T}} = \alpha^s \mathbf{M}^s \frac{\partial \boldsymbol{\mu}^*}{\partial \mathbf{T}}$. Second, the effects of $\mathbf{T}$ on utility $\mathbf{U}^*$ and welfare W^* are similar to those of $\boldsymbol{\mu}^*$.

To better understand these results, we will now run some simulations for the two applications in Section 2.2.

4.2 Applications and simulations

Let us illustrate Theorem 2 with our two main applications in Section 2.2 with five agents ($\mathcal{N} = \{1, 2, 3, 4, 5\}$) and three layers ($\mathcal{S} = \{1, 2, 3\}$). For all simulations, we denote $q_i^s = v_i^s + x_i^s + \phi^s \sum_{j \in \mathcal{N}} g_{ij}^s x_j^s$, for $i = 1, \cdots, 5$ and $s = 1, 2, 3$. We assume initially that $v_i^s = 0$, for $i = 1, \cdots, 5$, $s = 1, 2, 3$, $\alpha^1 = \alpha^2 = \alpha^3 = 1/3$, and $\mathbf{T} = (2, 2, 2, 2, 2)'$.

4.2.1 Management of multiple social relationships

Consider application (i) in Section 2.2, such that each agent, given their time constraint, has to allocate their efforts among the three layers. The three layers are as follows. Layer 1 (with adjacency matrix $\mathbf{G}^1$) is a star network in which player 1 is the star/center, layer 2 (with adjacency matrix $\mathbf{G}^2$) is a circle network in which each player has two direct links, and layer 3 (with adjacency matrix $\mathbf{G}^3$) is the complete network. See Figure 2.

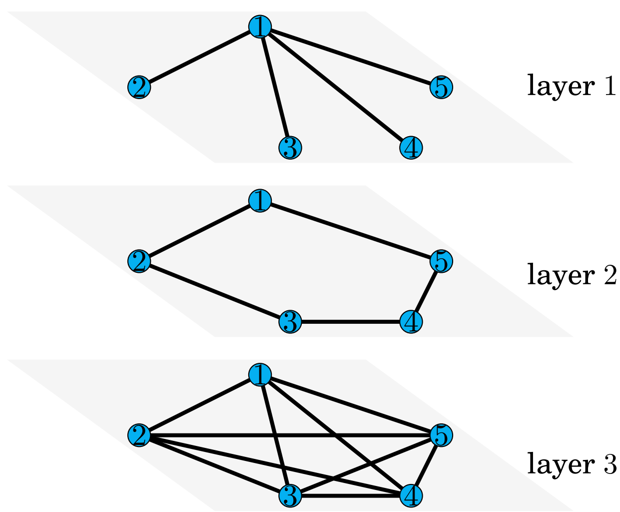

Figure 2: An example of multiple social relationships with three different layers.

Case 1: Strategic complementarities: Assume that $\phi^1 = \phi^2 = \phi^3 = -0.2$.[21] Remember that when ϕ^s is negative, efforts between two connected players i and j are strategic complements (i.e., the higher is j's effort, the higher is the increase in i's

[21] We can verify that Assumption 1 holds in this case. Furthermore, the equilibrium is unique and interior. The same comment applies to all other simulations.

effort on its marginal utility) but each individual exerts a negative spillover on the other (i.e., the higher is j's effort, the lower is the utility of i).

We obtain the following equilibrium values:

$$\mathbf{x}^* = \begin{pmatrix} 0.471 & 0.356 & 1.174 \\ 0.353 & 0.415 & 1.232 \\ 0.349 & 0.423 & 1.228 \\ 0.349 & 0.423 & 1.228 \\ 0.353 & 0.415 & 1.232 \end{pmatrix}, \quad \mathbf{q}^* = \begin{pmatrix} 0.190 & 0.190 & 0.190 \\ 0.259 & 0.259 & 0.259 \\ 0.255 & 0.255 & 0.255 \\ 0.255 & 0.255 & 0.255 \\ 0.259 & 0.259 & 0.259 \end{pmatrix}, \quad \boldsymbol{\mu}^* = \begin{pmatrix} 0.569 \\ 0.778 \\ 0.765 \\ 0.765 \\ 0.778 \end{pmatrix}, \quad \mathbf{U}^* = \begin{pmatrix} -1.662 \\ -1.350 \\ -1.366 \\ -1.366 \\ -1.350 \end{pmatrix}.$$

The aggregate welfare is equal to $W^* = -7.094$. Observe that in $\mathbf{x}^*$, columns correspond to layers and rows to players (there are five rows/players and three columns/layers). For example, the first column corresponds to layer 1, that is, $\mathbf{x}^{1*} = \left(x_1^{1*}, x_2^{1*}, \cdots x_5^{1*}\right)'$. The same interpretation holds for $\mathbf{q}^*$.

We see that, quite naturally, because of their location (the star), player 1 makes the highest effort in layer 1 but reduces their efforts in layers 2 and 3 (due to their time constraint $T_1 = 2$). Interestingly, even though players 2 and 5 have the same positions as players 3 and 4 in layer 1, they exert higher effort because they are directly linked to player 1 in layer 2 (strategic complentarities) while players 3 and 4 are not (in layer 3, they all have the same positions). However, if we look at effective equilibrium efforts $\mathbf{q}^*$, which captures own effort plus that of connected agents' efforts, player 1 has the lowest q_i^s in all layers because of the negative spillovers exerted by their neighbors. In other words, contrary to the standard theory of strategic complementarities in network games (Ballester et al., 2006), even though player 1 has a locational advantage in layer 1 and no locational disadvantage in the other layers, they end up having the lowest total effort q_i^s because of the negative spillovers from their neighbors. Thus, agent 1 obtains the lowest utility among the five players: $U_1^* < U_3^* = U_4^* < U_2^* = U_5^*$.[22]

Consider now the policy implications. First, we study the marginal effects on

[22] Agents 3 and 4 as well as agents 2 and 5 have symmetric positions.

increasing subsidies to all players on layer 1,[23] that is, $\mathbf{v}^1$.

$$\frac{\partial \mathbf{x^{1*}}}{\partial \mathbf{v}^1} = \begin{pmatrix} -0.818 & -0.199 & -0.177 & -0.177 & -0.199 \\ -0.199 & -0.728 & -0.097 & -0.075 & -0.079 \\ -0.177 & -0.097 & -0.721 & -0.094 & -0.075 \\ -0.177 & -0.075 & -0.094 & -0.721 & -0.097 \\ -0.199 & -0.079 & -0.075 & -0.097 & -0.728 \end{pmatrix}, \quad \frac{\partial \mathbf{U}^*}{\partial \mathbf{v}^1} = \begin{pmatrix} 1.751 & -0.016 & 0.109 & 0.109 & -0.016 \\ -0.136 & 1.201 & -0.239 & -0.153 & -0.149 \\ -0.052 & -0.226 & 1.231 & -0.229 & -0.138 \\ -0.052 & -0.138 & -0.229 & 1.231 & -0.226 \\ -0.136 & -0.149 & -0.153 & -0.239 & 1.201 \end{pmatrix}, \tag{39}$$

and

$$\frac{\partial W^*}{\partial \mathbf{v}^1} = \begin{pmatrix} 1.376 & 0.671 & 0.719 & 0.719 & 0.671 \end{pmatrix}. \tag{40}$$

In $\frac{\partial \mathbf{x^{1*}}}{\partial \mathbf{v}^1}$, we only look at the effect of a change in the subsidy in layer 1 (from $v_i^1 = 0$ to a positive value) on the efforts on layer 1, so that each row i corresponds to player i in layer 1 while each column j corresponds to the subsidy of player j in layer 1. For example, the second row ($i = 2$) third column ($j = 3$) corresponds to $\frac{\partial x_2^{1*}}{\partial v_3^1} = -0.097$. The same is true for $\frac{\partial \mathbf{U}^*}{\partial \mathbf{v}^1}$. Increasing the subsidy of player i in layer 1 (i.e., v_i^1) always reduces the effort v_i^1 of any player i in layer 1 because v_i^1 and x_i^1 are strategic substitutes (see (14)). This explains the negative signs on the diagonals of $\frac{\partial \mathbf{x^{1*}}}{\partial \mathbf{v}^1}$ in (39). The off diagonals capture the total effect of v_i^1 on x_j^1 with $i \neq j$, which are also negative due to strategic complementarity between effort x_i^1 and x_j^1 (recall $\phi^1 < 0$). For example, if v_3^1 increases, individual 3 decreases their effort x_3^{1*} due to the strategic substitutability between v_3^1 and x_3^1, which leads to a decrease in efforts of all other agents in layer 1 (that is, the efforts of agents 1, 2, 4, and 5 in layer 1) because of strategic complementarities in efforts due to $\phi^1 = -0.2 < 0$.

Concerning the impact of $\mathbf{v}^1$ on $\mathbf{U}^*$, the values in the diagonal are now positive, which means that a subsidy to agent i is always beneficial for this agent (see (1)). The off diagonals capture the effects of v_i^1 on the utility of agent j, $j \neq i$, which, in this case, can be positive or negative. For instance, $\frac{\partial U_1^*}{\partial v_2^1} = -0.016$, while $\frac{\partial U_1^*}{\partial v_3^1} = 0.109$. This is partially due to the different positions taken by agents 2 and 3 relative to agent 1: agent 2 is linked to agent 1 in all layers, while agent 3 is not linked to agent 1 on layer 2. Thus, when v_2^1 increases, x_2^{1*} decreases, which creates a positive externality on agent 1 on layer 1. However, because x_2^{2*} increases, it has a negative externality on player 1 on layer 2. The net effect of v_2^1 on the equilibrium payoff U_1^* is negative. With the same reasoning, we can understand why the effect of v_3^1 on the utility of player 1 is positive. This is because agent 3 is not directly linked to agent 1 on layer 2 and thus does not exert a negative externality on agent 1 on layer 2. Observe that if we had only layer 1 (monolayer), then the effects of v_2^1 on U_1^* would be positive.

Now, we study the marginal effects of increasing $\mathbf{T}$ on all players in layer 1. We obtain:

[23] For the sake of the presentation, in this section, we only study the policy implications of increasing subsidies or income on layer 1, since the effects on the other two layers are similar and have the same intuition. The simulation results of the impact on layers 2 and 3 are available upon request.

$$\frac{\partial \mathbf{x}^{\mathbf{1}*}}{\partial \mathbf{T}} = \begin{pmatrix} 0.332 & -0.035 & -0.013 & -0.013 & -0.035 \\ -0.003 & 0.311 & -0.058 & -0.035 & -0.039 \\ 0.021 & -0.062 & 0.314 & -0.058 & -0.040 \\ 0.021 & -0.040 & -0.058 & 0.314 & -0.062 \\ -0.003 & -0.039 & -0.035 & -0.058 & 0.311 \end{pmatrix}, \quad \frac{\partial \mathbf{U}^*}{\partial \mathbf{T}} = \begin{pmatrix} 1.714 & -0.366 & -0.241 & -0.241 & -0.366 \\ -0.268 & 1.228 & -0.212 & -0.126 & -0.122 \\ -0.179 & -0.216 & 1.241 & -0.219 & -0.128 \\ -0.179 & -0.128 & -0.219 & 1.241 & -0.216 \\ -0.268 & -0.122 & -0.126 & -0.212 & 1.228 \end{pmatrix}, \tag{41}$$

and

$$\frac{\partial W^*}{\partial \mathbf{T}} = \begin{pmatrix} 0.820 & 0.396 & 0.444 & 0.444 & 0.396 \end{pmatrix}. \tag{42}$$

When T_i increases, agent i has more time and thus increases their efforts on all layers; therefore, U_i^* increases (see the diagonals of $\frac{\partial \mathbf{x}^{\mathbf{1}*}}{\partial \mathbf{T}}$ and $\frac{\partial \mathbf{U}^*}{\partial \mathbf{T}}$). This implies that U_j^* decreases for $j \neq i$ due to negative spillovers (see the off-diagonals of $\frac{\partial \mathbf{U}^*}{\partial \mathbf{T}}$), while the efforts of agents j decrease or increase depending on their positions in the different layers (see the off-diagonals of $\frac{\partial \mathbf{x}^{\mathbf{1}*}}{\partial \mathbf{T}}$).

Case 2: Strategic substitutes: Assume now that $\phi^1 = 0.31$, $\phi^2 = 0.25$, $\phi^3 = 0.25$. Remember that when ϕ^s is positive, efforts between two connected players i and j are strategic substitutes (that is, the higher is j's effort, the lower is an increase in i's effort on their marginal utility) but each individual exerts a positive spillover on the other (that is, the higher is j's effort, the higher is the utility of i). We assume that the intensity of spillover effects in layer 1 is higher than that in layers 2 and 3 (i.e., $\phi^1 > \phi^2 = \phi^3$).

In this case, we obtain the following equilibrium values:

$$\mathbf{x}^* = \begin{pmatrix} 0.131 & 1.008 & 0.861 \\ 0.958 & 0.582 & 0.460 \\ 0.926 & 0.657 & 0.417 \\ 0.926 & 0.657 & 0.417 \\ 0.958 & 0.582 & 0.460 \end{pmatrix}, \quad \mathbf{q}^* = \begin{pmatrix} 1.299 & 1.299 & 1.299 \\ 0.999 & 0.999 & 0.999 \\ 0.967 & 0.967 & 0.967 \\ 0.967 & 0.967 & 0.967 \\ 0.999 & 0.999 & 0.999 \end{pmatrix}, \quad \boldsymbol{\mu}^* = \begin{pmatrix} 3.898 \\ 2.996 \\ 2.900 \\ 2.900 \\ 2.996 \end{pmatrix}, \quad \mathbf{U}^* = \begin{pmatrix} 0.262 \\ -0.001 \\ -0.034 \\ -0.034 \\ -0.001 \end{pmatrix}.$$

The aggregate welfare is given by $W^* = 0.191$.

Compared to case 1 with strategic complementarity, we obtain the reverse results. Player 1 has the lowest effort on layer 1 (free-riding) and the highest on layers 2 and 3. Furthermore, player 1 has the highest equilibrium utility, i.e., $U_1^* > U_2^* > U_3^*$. The intuition is similar but reverse to that of case 1 since efforts are now strategic substitutes but generate positive externalities on the equilibrium utility of the other agents.

Let us now examine the policy implications. First, we study the impact of increasing the subsidies on all players in layer 1. We obtain

$$\frac{\partial \mathbf{x}^{\mathbf{1}*}}{\partial \mathbf{v}^1} = \begin{pmatrix} -0.897 & 0.227 & 0.183 & 0.183 & 0.227 \\ 0.227 & -0.737 & 0.006 & -0.023 & -0.032 \\ 0.183 & 0.006 & -0.720 & 0.014 & -0.023 \\ 0.183 & -0.023 & 0.014 & -0.720 & 0.006 \\ 0.227 & -0.032 & -0.023 & 0.006 & -0.737 \end{pmatrix}, \quad \frac{\partial \mathbf{U}^*}{\partial \mathbf{v}^1} = \begin{pmatrix} 0.275 & -0.013 & -0.031 & -0.031 & -0.013 \\ -0.051 & 0.334 & 0.062 & 0.034 & 0.039 \\ -0.098 & 0.078 & 0.349 & 0.073 & 0.049 \\ -0.098 & 0.049 & 0.073 & 0.349 & 0.078 \\ -0.051 & 0.039 & 0.034 & 0.062 & 0.334 \end{pmatrix}, \tag{43}$$

and

$$\frac{\partial W^*}{\partial \mathbf{v}^1} = \begin{pmatrix} -0.023 & 0.488 & 0.487 & 0.487 & 0.488 \end{pmatrix}. \tag{44}$$

Giving a positive subsidy to any player but 1 leads to a decrease in U_1^* because agent 1, who is connected to everyone on layer 1, obtains lower spillovers from the other players. On the contrary, increasing the subsidy of any player but 1 always has a positive impact on the utility of all players but 1. For example, when v_2^1 increases, player 2 reduces their effort, which only hurts player 1 on layer 1. This, in turn, implies that player 2 increases their efforts on the other two layers, which create positive externalities on all other players. For player 1, the negative impact of increasing v_2^1 on layer 1 outweighs the positive impact of v_2^1 on the other layers. Thus, $\frac{\partial W_1^*}{\partial T_1} < 0$ and $\frac{\partial W_i^*}{\partial \mathbf{T}} > 0$ for all other players.

Now, we study the effect of the marginal increase in income $\mathbf{T}$ on all players in layer 1.

$$\frac{\partial \mathbf{x}^{\mathbf{1}*}}{\partial \mathbf{T}} = \begin{pmatrix} 0.357 & -0.051 & -0.095 & -0.095 & -0.051 \\ -0.017 & 0.334 & 0.076 & 0.048 & 0.039 \\ -0.041 & 0.062 & 0.337 & 0.070 & 0.034 \\ -0.041 & 0.034 & 0.070 & 0.337 & 0.062 \\ -0.017 & 0.039 & 0.048 & 0.076 & 0.334 \end{pmatrix}, \quad \frac{\partial \mathbf{U}^*}{\partial \mathbf{T}} = \begin{pmatrix} 0.248 & 0.072 & 0.054 & 0.054 & 0.072 \\ 0.094 & 0.318 & 0.047 & 0.018 & 0.023 \\ 0.072 & 0.048 & 0.318 & 0.042 & 0.019 \\ 0.072 & 0.019 & 0.042 & 0.318 & 0.048 \\ 0.094 & 0.023 & 0.018 & 0.047 & 0.318 \end{pmatrix}, \tag{45}$$

and

$$\frac{\partial W^*}{\partial \mathbf{T}} = \begin{pmatrix} 0.581 & 0.480 & 0.479 & 0.479 & 0.480 \end{pmatrix}. \tag{46}$$

When $\mathbf{T}$ increases, all agents have more income and all agents benefit; thus, $\frac{\partial \mathbf{U}^*}{\partial \mathbf{T}} > \mathbf{0}$ and $\frac{\partial \mathbf{W}^*}{\partial \mathbf{T}} > \mathbf{0}$. Moreover, the effect of an increase in T_i on the effort of agent i in layer 1, i.e., x_i^1, is always positive (see the diagonal entries of $\frac{\partial \mathbf{x}^{\mathbf{1}*}}{\partial \mathbf{T}}$). The effect of T_i on x_j^1, $i \neq j$, can be positive or negative depending on how the agents reallocate their efforts on layers 2 and 3 (see the off-diagonals of $\frac{\partial \mathbf{x}^{\mathbf{1}*}}{\partial \mathbf{T}}$).

4.2.2 Multiple public goods

Consider now application (ii) in Section 2.2. Assume that $\phi^1 = 0.2$, $\phi^2 = 0$, $\phi^3 = 1$. The three layers are as follows. Layer 1 (with adjacency matrix $\mathbf{G}^1$) is a star network in which player 1 is the star, layer 2 is the empty network, and layer 3 (with adjacency matrix $\mathbf{G}^3$) is the complete network. Observe that apart from player 1, all other players have the same positions in all layers. See Figure 3.

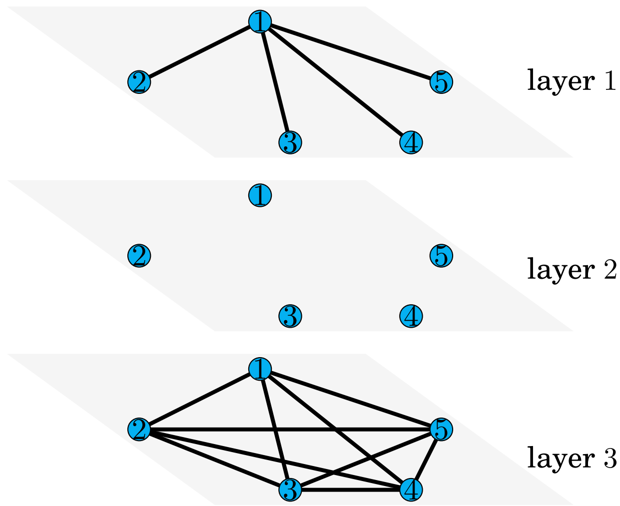

Figure 3: An example of multiple public goods with three different layers.

In this case, we obtain the following equilibrium values:

$$\mathbf{x}^* = \begin{pmatrix} 0.237 & 0.995 & 0.768 \\ 0.948 & 0.995 & 0.057 \\ 0.948 & 0.995 & 0.057 \\ 0.948 & 0.995 & 0.057 \\ 0.948 & 0.995 & 0.057 \end{pmatrix}, \quad \mathbf{q}^* = \begin{pmatrix} 0.995 & 0.995 & 0.995 \\ 0.995 & 0.995 & 0.995 \\ 0.995 & 0.995 & 0.995 \\ 0.995 & 0.995 & 0.995 \\ 0.995 & 0.995 & 0.995 \end{pmatrix}, \quad \boldsymbol{\mu}^* = \begin{pmatrix} 2.986 \\ 2.986 \\ 2.986 \\ 2.986 \\ 2.986 \end{pmatrix}, \quad \mathbf{U}^* = \begin{pmatrix} -0.005 \\ -0.005 \\ -0.005 \\ -0.005 \\ -0.005 \end{pmatrix}.$$

The aggregate welfare is $W^* = -0.024$.

Player 1, who is the star, free-rides on layer 1 (local public good) by providing the lowest effort but compensates this by making the highest effort on layer 3 (global public good). As shown in Section 3.3.2, all agents exert the same effort on layer 2 (private good) and obtain the same utility, i.e., $U_1^* = \cdots = U_5^*$. The same is true for $\mathbf{q}^*$ and $\boldsymbol{\mu}^*$.

Let us now examine the policy implications. First, we study the marginal effect of increasing the subsidies $\mathbf{v}^1$ to all players in layer 1. We get

$$\frac{\partial \mathbf{x}^{\mathbf{1}*}}{\partial \mathbf{v}^1} = \begin{pmatrix} -1.185 & 0.261 & 0.261 & 0.261 & 0.261 \\ 0.261 & -0.957 & 0.043 & 0.043 & 0.043 \\ 0.261 & 0.043 & -0.957 & 0.043 & 0.043 \\ 0.261 & 0.043 & 0.043 & -0.957 & 0.043 \\ 0.261 & 0.043 & 0.043 & 0.043 & -0.957 \end{pmatrix}, \quad \frac{\partial \mathbf{U}^*}{\partial \mathbf{v}^1} = \begin{pmatrix} 0.024 & 0.095 & 0.095 & 0.095 & 0.095 \\ 0.024 & 0.095 & 0.095 & 0.095 & 0.095 \\ 0.024 & 0.095 & 0.095 & 0.095 & 0.095 \\ 0.024 & 0.095 & 0.095 & 0.095 & 0.095 \\ 0.024 & 0.095 & 0.095 & 0.095 & 0.095 \end{pmatrix}, \tag{47}$$

and

$$\frac{\partial W^*}{\partial \mathbf{v}^1} = \begin{pmatrix} 0.119 & 0.476 & 0.476 & 0.476 & 0.476 \end{pmatrix}. \tag{48}$$

First, due to the substitutability between x_i^{1*} and v_i^1, the diagonal of $\frac{\partial \mathbf{x}^{1*}}{\partial \mathbf{v}^1}$ is always negative. On the contrary, the off-diagonal of $\frac{\partial \mathbf{x}^{1*}}{\partial \mathbf{v}^1}$ is always positive because of strategic substitutability in efforts. Second, increasing the subsidy of a player i always leads to the same utility increase for all agents because the equilibrium utility is the same for everyone (see each row of $\frac{\partial \mathbf{U}^*}{\partial \mathbf{v}^1}$). Observe that an increase in v_1^1 (first row) has a lower impact on the equilibrium utility of all agents than an increase of v_j^1 for $j \neq 1$ (second to fifth row) because player 1 has the lowest effort on layer 1.

Next, we study the marginal increase in income $\mathbf{T}$ on all players in layer 1. We have

$$\frac{\partial \mathbf{x}^{1*}}{\partial \mathbf{T}} = \begin{pmatrix} 0.024 & 0.024 & 0.024 & 0.024 & 0.024 \\ 0.095 & 0.095 & 0.095 & 0.095 & 0.095 \\ 0.095 & 0.095 & 0.095 & 0.095 & 0.095 \\ 0.095 & 0.095 & 0.095 & 0.095 & 0.095 \\ 0.095 & 0.095 & 0.095 & 0.095 & 0.095 \end{pmatrix}, \quad \frac{\partial \mathbf{U}^*}{\partial \mathbf{T}} = \begin{pmatrix} 0.100 & 0.100 & 0.100 & 0.100 & 0.100 \\ 0.100 & 0.100 & 0.100 & 0.100 & 0.100 \\ 0.100 & 0.100 & 0.100 & 0.100 & 0.100 \\ 0.100 & 0.100 & 0.100 & 0.100 & 0.100 \\ 0.100 & 0.100 & 0.100 & 0.100 & 0.100 \end{pmatrix}, \tag{49}$$

and

$$\frac{\partial W^*}{\partial \mathbf{T}} = \begin{pmatrix} 0.500 & 0.500 & 0.500 & 0.500 & 0.500 \end{pmatrix}. \tag{50}$$

First, increasing T_i for agent i has the same impact on equilibrium utility as increasing T_j for agent j due to the neutrality result established in Section 3.3.2. Second, the effects of increasing T_i on the efforts of all agents are different for player 1 (see the first row of $\frac{\partial \mathbf{x}^{1*}}{\partial \mathbf{T}}$) than for the other players (see the other rows of $\frac{\partial \mathbf{x}^{1*}}{\partial \mathbf{T}}$) because player 1 has a more central position on layer 1 and thus makes less effort on this layer.

4.3 Targeting problems

Suppose the social planner can design interventions to maximize aggregate welfare W^*. Define a row vector $\boldsymbol{\eta} = (\eta_1, \cdots, \eta_n) := \frac{\partial W^*}{\partial \mathbf{T}}$ as the effect of marginal changes in $\mathbf{T}$ on the aggregate welfare.[24] By Theorem 2,

$$\boldsymbol{\eta} = (1, \cdots, 1)\frac{\partial \mathbf{U}^*}{\partial \mathbf{T}} = \left(\sum_{s \in \mathcal{S}} \alpha^s\right) (1/\mu_1^*, \cdots, 1/\mu_n^*) \left(\sum_{t \in \mathcal{S}} \alpha^t \mathbf{M}^t\right)^{-1}. \tag{51}$$

[24] Here we focus on the marginal effects of interventions. Another valuable venue is to analyze the social planner's constrained targeting intervention program, along the lines of Galeotti et al. (2020). We leave this for future work.

Proposition 6.

(i) Consider the problem in which the planner can only allocate income T_i to a player i that marginally increases welfare. Then, the planner should target the player with the highest η_i.

(ii) Consider the problem in which the planner can only allocate subsidy v_i^s to a player i on layer s that marginally increases welfare. Then, the planner should target the player with the highest τ_i^s, where

$$\boldsymbol{\tau}^s = (1, \cdots, 1)\frac{\partial \mathbf{U}^*}{\partial \mathbf{v}^s} = \boldsymbol{\eta}\mathbf{M}^s. \tag{52}$$

Let us provide some intuition of the targeting results shown in Proposition 6. Consider case (i). The maximization problem can be formulated as

$$\max_{i \in \mathcal{N}} \frac{\partial W^*}{\partial T_i} = \frac{\partial \sum_{j \in \mathcal{N}} U_j^*}{\partial T_i}, \tag{53}$$

which is equivalent to choosing player i with the highest η_i.

Now, consider case (ii). The planner needs to solve the following problem

$$\max_{i \in \mathcal{N}, s \in \mathcal{S}} \frac{\partial W^*}{\partial v_i^s} = \frac{\partial \sum_{j \in \mathcal{N}} U_j^*}{\partial v_i^s}. \tag{54}$$

The solution of this problem can be obtained in two steps. First, for each layer s, we can derive the *key player* on that layer by solving the following problem: $\max_{i \in \mathcal{N}} \tau_i^s$. In the second step, we optimize over s.

Compared to the standard targeting intervention problems analyzed in monolayer networks (Ballester et al., 2006; Galeotti et al., 2020; Kor and Zhou, 2022), the planner now needs to take into account both the effect on the current layer and that on the other layers. One can see from $\boldsymbol{\eta}$ (see (51)) and $\boldsymbol{\tau}^s$ (see (52)) that these indices depend on $\mathbf{M}^s$, the matrix of interactions between agents in a given layer s, as well as on the interaction between layers through $\boldsymbol{\mu}^*$.

Let us now illustrate these results with our numerical simulations.

Consider first case 1 with strategic complementarities and negative spillovers ($\phi^1 = \phi^2 = \phi^3 = -0.2$). The player with the highest η_i is player 1 (see (42)), and the player with the highest τ_i^1 is also player 1 (see (40)). This is because the direct impact of T_1 or v_1^1 is the highest one on player 1 because player 1 has the lowest utility and the indirect negative spillovers that player 1 exerts on all other players are relatively small.

Consider now case 2 with strategic substitutes and positive spillovers ($\phi^1 = 0.31$, $\phi^2 = 0.25$, $\phi^3 = 0.25$). The player with the highest η_i is again player 1 (see (46)), while the player with the highest τ_i^1 is not player 1 (see (44)) but player 2 or 5. Indeed, for the impact of T_1, the direct effect on player 1 is smaller compared to the other players, but the indirect positive spillover effects on all other players are much larger. Therefore, the planner should target player 1 in terms of income T_i. This is not true for the subsidy policy, in which case the planner should not target player 1. Indeed, when the planner gives the subsidy v_1^1 to player 1, only player 1 benefits in terms of utility while all the other players experience a decrease in their utility (see the first row of $\frac{\partial \mathbf{U}^*}{\partial \mathbf{v}^1}$ in (43)). On the contrary, when the planner gives the subsidy to any player but 1, the reverse is true; that is, all agents see an increase in their utility while player 1 experiences a decrease (see rows second to fifth of $\frac{\partial \mathbf{U}^*}{\partial \mathbf{v}^1}$ in (43)). Because the former positive effect is smaller than the latter negative effect, the planner needs to target any player but 1.

Finally, consider the case with multiple public goods ($\phi^1 = 0.2$, $\phi^2 = 0$, $\phi^3 = 1$). For the income policy T_i, due to neutrality, it does not matter whom the planner targets since $\eta_1 = \cdots = \eta_5$ (see (50)). For the subsidy policy, the planner should not target player 1 (see (48)). By comparing the first and the other rows of $\frac{\partial \mathbf{U}^*}{\partial \mathbf{v}^1}$ in (47), we see that both the direct and indirect effects are much smaller when targeting player 1 rather than anyone else (see (21)).

5 Conclusion

Multiplex networks are a powerful tool for understanding and analyzing complex systems. They allow us to capture the richness and complexity of real-world relationships and interactions, and they provide a framework for developing new insights and understanding into a wide range of phenomena. In this paper, by using multiplex networks, we capture the rich complexity of social interactions, which lead to new insights and understanding of social phenomena.

We develop a simple model of network multiplexity and establish the equilibrium properties of this model using best-reply potential techniques. We characterize the unique Nash equilibrium and show that efforts depend on the specificity of the layer, the intensity of spillovers, and the individual position in each layer. We perform some comparative statics exercises by studying within-layer and cross-layer shock propagation. We also examine targeted interventions such as finding the key player in a multilayer network in which a planner takes into account both the impact of its policy on a given layer and that on the other interconnected layers.

Different extensions are possible. First, we consider discrete (Leister et al., 2022) instead of continuous actions and examine how this changes our results. Second, we extend our model to large population (or graphon) games (Parise and Ozdaglar, 2023) and derive new results for multilayer networks. We leave these exciting topics for future research.

Appendix

A Proofs

Proof of Theorem 1: The second part immediately follows from the first part. So it suffices to establish (10). To this end, we need the following Lemma:

Lemma 1. *Fixing parameters $v^s > 0, \forall s$. Define a set Δ as the set of nonnegative vectors $\mathbf{x} = (x^1, \cdots, x^s) \geq \mathbf{0}$ satisfying $\sum_s x^s = T > 0$. Consider two optimization problems (P1) and (P2), where*

$$(P1) \quad \max_{\mathbf{x}\in\Delta} \sum_{s\in\mathcal{S}} \alpha^s \ln(v^s + x^s) \tag{55}$$

and

$$(P2) \quad \max_{\mathbf{x}\in\Delta} - \sum_{s\in\mathcal{S}} \frac{1}{2\alpha^s}(v^s + x^s)^2. \tag{56}$$

The solution to (P1) coincides with that to (P2).

To show Lemma 1, we note that in each problem (Pi), $i = 1, 2$, the objective function is strictly concave, and the constraint set is compact and convex. Therefore, for each problem (Pi), $i = 1, 2$, a unique solution exists, which must satisfy the local necessary conditions.

The Lagrange to the first problem (P1) is

$$\mathcal{L} = \sum_{s\in\mathcal{S}} \alpha^s \ln(v^s + x^s) - \lambda(\sum_s x^s - T)$$

where $\lambda > 0$ is the multiplier to the constraint $\sum_s x^s = T > 0$. The first-order condition (FOC) with respect to x^s yields

$$\frac{\alpha^s}{(v^s + x^s)} - \lambda \leq 0 \, (= 0 \text{ if } x^s > 0). \tag{57}$$

The unique solution $\mathbf{x}^*$ to (P1) is the unique solution to the FOCs (57) (recall that $\mathbf{x}^*$ is in Δ).

Similarly, the Lagrange to the second problem (P2) is

$$\mathcal{L} = - \sum_{s\in\mathcal{S}} \frac{1}{2\alpha^s}(v^s + x^s)^2 - \mu(\sum_s x^s - T)$$

where $\mu < 0$ is the multiplier to the constraint $\sum_s x^s = T > 0$. The FOC with respect to x^s yields

$$- \frac{(v^s + x^s)}{\alpha^s} - \mu \leq 0 \, (= 0 \text{ if } x^s > 0). \tag{58}$$

We claim that $(\mathbf{x}^*, \lambda^*)$ solves the corresponding FOCs (57) of problem (P1) if and only if $(\mathbf{x}^*, \mu^* = -1/\lambda^*)$ solves the corresponding FOCs (58) of problem (P2). This immediately implies

$$\frac{\alpha^s}{(v^s + x^s)} \leq \lambda \Longleftrightarrow \frac{(v^s + x^s)}{\alpha^s} \geq \frac{1}{\lambda} \Longleftrightarrow - \frac{(v^s + x^s)}{\alpha^s} - \left(-\frac{1}{\lambda}\right) \leq 0.$$

To show (10), we note that, given $\mathbf{x}_{-i} \in K_{-i}$, agent i's utility is strictly concave in $\mathbf{x}_i$ with

$$\frac{\partial U_i}{\partial x_i^s} = \frac{\alpha^s}{(v_i^s + x_i^s + \phi^s \sum_{j\in\mathcal{N}} g_{ij}^s x_j^s)}, \forall s \in S. \tag{59}$$

Due to strict concavity, agent i's best response $\arg\max_{\mathbf{x}_i \in K_i} U_i(\mathbf{x}_i, \mathbf{x}_{-i})$ is unique, and it satisfies the following FOCs:

$$\frac{\alpha^s}{(v_i^s + x_i^s + \phi^s \sum_{j\in\mathcal{N}} g_{ij}^s x_j^s)} - \lambda_i \leq 0\,(= 0 \text{ if } x_i^s > 0), \tag{60}$$

where λ_i is the multiplier to i's constraint $\sum_{s\in\mathcal{S}} x_i^s = T_i$. Furthermore, given $\mathbf{x}_{-i}$, $\theta(\cdot, \mathbf{x}_{-i})$ is quadratic in $\mathbf{x}_i$ with

$$-\frac{\partial \theta}{\partial x_i^s} = \frac{(v_i^s + x_i^s + \phi^s \sum_{j\in\mathcal{N}} g_{ij}^s x_j^s)}{\alpha^s}, \forall s \in S \tag{61}$$

and a negative definite Hessian matrix $-(\frac{\partial^2 \theta}{\partial^2 x_i^{s'} x_i^{s''}})_{1\leq s',s''\leq s} = -diag(1/\alpha_1, \cdots, 1/\alpha_s)$. Hence, $\theta(\cdot, \mathbf{x}_{-i})$ is strictly concave in $\mathbf{x}_i$, and thus, $\arg\max_{\mathbf{x}_i\in K_i} \theta(\mathbf{x}_i, \mathbf{x}_{-i})$ is unique and must satisfy the following FOCs:

$$-\frac{(v_i^s + x_i^s + \phi^s \sum_{j\in\mathcal{N}} g_{ij}^s x_j^s)}{\alpha^s} - \mu_i \leq 0\,(= 0 \text{ if } x_i^s > 0), \tag{62}$$

where μ_i is the multiplier to i's constraint: $\sum_{s\in\mathcal{S}} x_i^s = T_i$. The rest just follows from the argument behind the proof of Lemma 1 and the construction of θ. □

Proof of Proposition 1: By Theorem 1, $\mathbf{X}^* = (\mathbf{x}_1^*, \cdots, \mathbf{x}_n^*)$ is an equilibrium if and only if, for any $i \in \mathcal{N}$,

$$\theta(\mathbf{x}_1^*, \mathbf{x}_{-i}^*) \geq \theta(\mathbf{x}_1, \mathbf{x}_{-i}^*), \forall \mathbf{x}_i \in K_i.$$

Since θ is strictly concave in $\mathbf{x}_i$, for each fixed $\mathbf{x}_{-i}$, we can replace the above optimality condition using the corresponding FOCs and then summarize them over all i as a solution to the following variational inequality:

$$\langle -\nabla\theta(\mathbf{X}^*), \mathbf{X} - \mathbf{X}^* \rangle \geq 0, \ \ \forall \mathbf{X} \in \prod_{i\in\mathcal{N}} K_i. \tag{63}$$

Here $\nabla\theta$ is the gradient of θ. It is straightforward to check that Assumption 1 is equivalent to the fact that $-\nabla\theta$ is a monotone operator on K, which immediately implies the uniqueness of the solution to VI in (63) (see Nagurney (1999)).[25] So Proposition 1(i) is proved.

[25] To see the intuition, recall that θ is a quadratic function defined on a convex compact set K. The quadratic form

$$Q(\mathbf{z}^1, \cdots, \mathbf{z}^s) = \sum_{s\in\mathcal{S}} \left(\frac{1}{2\alpha^s}\right) (\mathbf{z}^s)'(\mathbf{I}_n + \phi^s \mathbf{G}^s)\mathbf{z}^s$$

is only the quadratic term of $-\theta$. Meanwhile, the linear constrains $\sum_{s\in\mathcal{S}} \mathbf{z}^s = \mathbf{0}$ are due to binding budget constraints for agents. Consequently, under Assumption 1, θ is strictly concave on K, implying the uniqueness of the global maximizer of θ on K.

Consider now the proof of Proposition 1(ii). When $1+\lambda_{\min}(\phi^s \mathbf{G}^s) > 0, \forall s \in \mathcal{S}$, the quadratic form Q is positive definite without the linear constraints. The rest just follows from Proposition 1(i). $\square$

Proof of Proposition 2: At an interior equilibrium $\mathbf{X}^*$, the FOCs of agent i yield

$$\frac{\alpha^s}{(v_i^s + x_i^{s*} + \phi^s \sum_{j\in\mathcal{N}} g_{ij}^s x_j^{s*})} = \lambda_i^*, \forall s \in S,$$

where λ_i equals the multiplier of i's resource constraint. As a result,

$$(v_i^s + x_i^{s*} + \phi^s \sum_{j\in\mathcal{N}} g_{ij}^s x_j^{s*}) = \alpha^s \underbrace{(1/\lambda_i^*)}_{:=\mu_i^*}, \tag{64}$$

implying that, for each layer s, $\mathbf{v}^s + \mathbf{x}^{s*} + \phi^s \mathbf{G}^s \mathbf{x}^{s*} = \alpha^s \boldsymbol{\mu}^*$, or, equivalently, $\mathbf{x}^{s^*} = [\mathbf{I}_n + \phi^s \mathbf{G}^s]^{-1}(\boldsymbol{\mu}^* - \alpha^s \mathbf{v}^s) = \mathbf{M}^s(\boldsymbol{\mu}^* - \alpha^s \mathbf{v}^s)$. Here $\boldsymbol{\mu}^* = (\mu_1^*, \cdots, \mu_n^*)'$. Using the constraints $\sum_{s\in\mathcal{S}} \mathbf{x}^{s^*} = \mathbf{T}$, we solve for $\boldsymbol{\mu}^*$ as given in Definition 1.

Part (ii) is obvious as the equilibrium payoff of agent i, using (64), is equal to $U_i^* = \sum_{s\in\mathcal{S}} \alpha^s \ln(\alpha^s \mu_i^*)$. Hence, we finish the proof. $\square$

Proof of Corollary 1: By Proposition 2(ii), $\ln U_i^* - \ln U_j^* = (\sum_{s\in\mathcal{S}} \alpha^s) \left(\ln(\mu_i^*) - \ln(\mu_j^*)\right)$. The result just follows. $\square$

Proof of Proposition 3: We have

$$\mathbf{M}^s = \mathbf{Z}\, diag\left(\frac{1}{1+\phi^s\gamma_1}, \cdots, \frac{1}{1+\phi^s\gamma_n}\right)\mathbf{Z}^{-1}, \text{for each } s \in \mathcal{S}.$$

This implies that $\mathbf{M}^s$ commutes with $\mathbf{M}^t$ for any s and t. Then,

$$\begin{aligned}
\widehat{\boldsymbol{\mu}}^* &= \mathbf{Z}^{-1}\boldsymbol{\mu}^* = \mathbf{Z}^{-1}\left(\sum_{s\in\mathcal{S}} \alpha^s \mathbf{M}^s\right)^{-1}\left(\mathbf{T} + \sum_{s\in\mathcal{S}} \mathbf{M}^s \mathbf{v}^s\right). \\
&= \left(\sum_{s\in\mathcal{S}} \alpha^s diag\left(\frac{1}{1+\phi^s\gamma_1}, \cdots, \frac{1}{1+\phi^s\gamma_n}\right)\right)^{-1}\left(\widehat{\mathbf{T}} + \sum_{s\in\mathcal{S}} diag\left(\frac{1}{1+\phi^s\gamma_1}, \cdots, \frac{1}{1+\phi^s\gamma_n}\right)\widehat{\mathbf{v}}^s\right).
\end{aligned}$$

In other words, in this eigenspace of $\mathbf{G}$, we have

$$\widehat{\mu}_i^* = \left(\sum_{s\in\mathcal{S}} \frac{\alpha^s}{1+\phi^s\gamma_i})\right)^{-1}\left(\widehat{T}_i + \sum_{s\in\mathcal{S}} \frac{1}{1+\phi^s\gamma_i}\widehat{v}^s\right),$$

which is equation (23) in Proposition 3.

Now, we can rewrite the equilibrium allocations in the eigenspace of $\mathbf{G}$ as follows:

$$\begin{aligned}
\widehat{\mathbf{x}}^{s*} &= \mathbf{Z}^{-1}\mathbf{x}^{s*} = \mathbf{Z}^{-1}\mathbf{M}^s(\alpha^s\boldsymbol{\mu}^* - \mathbf{v}^s), \\
&= diag\left(\frac{1}{1+\phi^s\gamma_1}, \cdots, \frac{1}{1+\phi^s\gamma_n}\right)\left(\alpha^s\widehat{\boldsymbol{\mu}}^* - \widehat{\mathbf{v}}^s\right),
\end{aligned}$$

Therefore,

$$\widehat{x}_i^{s*} = \frac{1}{1+\phi^s\gamma_i}(\alpha^s\widehat{\mu}_i^* - \widehat{v}_i^s),$$

which is (22) in Proposition 3. ☐

Proof of Theorem 2: It follows from the discussion after Theorem 2 in the main text. ☐

Proof of Proposition 4 : The results directly follow from (24) and (25). ☐

Proof of Proposition 5: An interior efficient allocation $\mathbf{X}^*$ must satisfy the corresponding FOCs with equalities, which lead to the system of equations in (27). Furthermore, W is strictly concave in $\mathbf{X}$. Thus, a solution to FOCs must be globally optimal. ☐

Proof of Proposition 6: It follows from the discussion in the main text. ☐

B Interiority of the equilibrium effort

Under Assumption 1, there exists a unique equilibrium. If, for each $s \in \mathcal{S}$, $\mathbf{x}^{s*}$ defined in (14) is interior, then $\mathbf{X}^* = (\mathbf{x}^{s*}, s \in \mathcal{S})$ must be a unique equilibrium by Proposition 2. To this end, we derive first-order approximations of the equilibrium objectives when the ϕ^ss are small.

For simplicity, assume $\mathbf{v}^s = \mathbf{0}$. In this case, we obtain the following Taylor expansions:

$$\begin{aligned}
\mathbf{M}^s &\approx \mathbf{I} - \phi^s\mathbf{G}^s, \\
\boldsymbol{\mu}^* &\approx \frac{1}{\bar{\alpha}}\left(\mathbf{I} + \sum_{t\in\mathcal{S}} \frac{\alpha^t}{\bar{\alpha}}\phi^t\mathbf{G}^t\right)\mathbf{T}, \\
\mathbf{x}^{s*} &\approx \frac{\alpha^s}{\bar{\alpha}}\left(\mathbf{I} - \phi^s\mathbf{G}^s + \sum_{t\in\mathcal{S}} \frac{\alpha^t}{\bar{\alpha}}\phi^t\mathbf{G}^t\right)\mathbf{T},
\end{aligned}$$

where $\bar{\alpha} = \sum_{s\in\mathcal{S}}\alpha^s > 0$. These approximations are correct up to quadratic or higher-order terms of ϕ^ss. When $\mathbf{T} = \mathbf{1}$, we can further simplify these expressions. For each $i \in \mathcal{N}$, we obtain

$$\mu_i^* \approx \frac{1}{\bar{\alpha}}\left(1 + \sum_{t\in\mathcal{S}} \frac{\alpha^t}{\bar{\alpha}}\phi^t d_i^t\right), \tag{65}$$

$$x_i^{s*} \approx \frac{\alpha^s}{\bar{\alpha}}\left(1 - \phi^s d_i^s + \sum_{t\in\mathcal{S}} \frac{\alpha^t}{\bar{\alpha}}\phi^t d_i^t\right), \tag{66}$$

where d_i^t is the degree of agent $i \in \mathcal{N}$ in layer $t \in \mathcal{S}$. These expressions are useful to see the effects of spillovers ϕ^s on the equilibrium payoff U_i^*, which has a one-to-one relationship to μ_i^* by Corollary 1, and equilibrium efforts.

In particular, when $\phi^s \approx 0$, for all s, from the above approximations, we see that $x_i^{s^*}$ and μ_i^* must all be positive. By continuity, this shows that the equilibrium efforts must be interior when the spillovers are not too large. In all the simulations, we verify that, indeed, the equilibrium efforts are always strictly positive (see Section 4.2).

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