WEB APPENDIX: Additional material and notes on "The Analytic Theory of a Monetary Shock"*

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Abstract

This document collects a set of notes, extensions and details of derivations for the paper "The Analytic Theory of a Monetary Shock"

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*Appendix to be posted online.

D A class of price setting models

We consider an economy with a representative household facing a continuum of monopolistic firms. Each firm price setting is subject to a fixed (menu) cost. We focus on a general equilibrium whose main features we sketch below.

Households. The preferences of the representative households are given by:

$$\int_0^\infty e^{-rt} \left[U\left(c(t)\right) - \alpha\, L(t) + \log\left(\frac{M(t)}{P(t)}\right)\right] dt \tag{1}$$

where $c(t)$ is an aggregate of the goods produced by all firms, $L(t)$ is the labor supply, $M(t)$ the nominal quantity of money, and $P(t)$ the nominal price of one unit of consumption, formally defined below (all variables at time t). We will use $U(c) = (c^{1-\epsilon} - 1)/(1-\epsilon)$. There is a unit mass of firms, index by $k \in [0,1]$, and each of them produces n goods, index by $i = 1, ..., n$. There is a preference shock $A_{k,i}(t)$ associated with good i produced by firm k at time t, which acts as a multiplicative shifter of the demand of each good i. Let $c_{k,i}(t)$ be the consumption of the product i produced by firm k at time t. The composite Dixit-Stiglitz consumption good c is

$$c(t) = \left[\int_0^1 \left(\sum_{i=1}^n A_{k,i}(t)^{\frac{1}{\eta}}\, c_{k,i}(t)^{\frac{\eta-1}{\eta}}\right) dk\right]^{\frac{\eta}{\eta-1}} \tag{2}$$

For firm k to produce $y_{k,i}(t)$ of the i good at time t requires $L_{k,i}(t) = y_{k,i}(t) Z_{k,i}(t)$ units of labor, so that $W(t) Z_{k,i}(t)$ is the marginal cost of production. We assume that $A_{k,i}(t) = Z_{k,i}(t)^{\eta-1}$ so the (log) of marginal cost and the demand shock are perfectly correlated.[1] We assume that $Z_{k,i}(t) = \exp\left(\sigma \mathcal{W}_{k,i}(t)\right)$ where $\mathcal{W}_{k,i}$ are standard BM's, independent across all i, k.

The budget constraint of the representative agent is

$$M(0) + \int_0^\infty Q(t) \left[\bar{\Pi}(t) + \tau(t) + (1+\tau_L) W(t) L(t) - R(t) M(t) - \int_0^1 \sum_{i=1} P_{k,i}(t) c_{k,i}(t) dk\right] dt = 0$$

where $R(t)$ is the nominal interest rates, $Q(t) = \exp\left(-\int_0^t R(s) ds\right)$ the price of a nominal bond, $W(t)$ the nominal wage, $\tau(t)$ the lump sum nominal transfers, τ_L a constant labor subsidy rate, and $\bar{\Pi}(t)$ the aggregate (net) nominal profits of firms.

[1] We assume that demand and cost shocks are correlated so that the relative demand of the different consumption goods is stationary, and in particular the frictionless profit can be made constant (see the profit equation (12) below). Furthermore, the constant relative demands obtained in the frictionless case allow for a simple analytical characterization of the aggregate price level whose weights on the different product varieties are constant (see equation (14) below). It it for these reasons that the assumption of correlated demand and cost shocks has been analyzed in the literature by several authors, e.g. Woodford (2009) (see his Section 3), Bonomo, Carvalho, and Garcia (2010) Midrigan (2011), so that exploring this benchmark case is useful.

Firms. We consider (a continuum of) profit maximizing firms, each producing and selling n products, with constant time discount factor r. The primal profit maximising problem is formally defined in Appendix E.2. We show there that for a small menu cost that problem is well approximated by a quadratic problem where each firm faces an n-dimensional state x that we refer to as the vector of price gaps. Each gap x_i measures the distance between the actual (log) price p_i and the optimal profit-maximising price p_i^*, so that $x_i \equiv p_i - p_i^*$. We first formally define the approximate problem and then discuss its interpretation.

Each element of the vector of price gaps x follows

$$x_i(t) = \sigma \mathcal{W}_i(t) + \sum_{j:\tau_j<t} \Delta p_i(\tau_j) \text{ for all } t \geq 0 \text{ and } i = 1, 2, ..., n, \tag{3}$$

$\Delta p_i(\tau_j) \equiv \lim_{t\downarrow\tau_j} x_i(t) - \lim_{t\uparrow\tau_j} x_i(t)$ and $x(0) = x$ where the τ_j are the (stopping) times at which control (i.e. a price change) is exercised. The n Brownian Motions (BM henceforth) are independent, so $\mathbb{E}[\mathcal{W}_i(t)\mathcal{W}_j(t')] = 0$ for all $t, t' \geq 0$ and $i, j = 1, ..., n$. Given a sequence of price gaps[2] $\{x(t)\}$ and the value of the current state $x(0) = x$, we can compute the wedge between the maximum profits of the firm and the profits under sticky price (an objective to be minimized) as

$$\mathbf{V}(\boldsymbol{\tau}, \boldsymbol{\Delta p}; x) \equiv \mathbb{E}\left[\sum_{j=1}^{\infty} e^{-r\tau_j} \psi\, \mathcal{I}_c(\tau_j) + \int_0^{\infty} e^{-rt} B \left(\sum_{i=1}^{n} x_i^2(t)\right) dt \,\middle|\, x(0) = x\right] \tag{4}$$

where ψ denotes the menu cost and B is a parameter. The indicator function $\mathcal{I}_c(\tau_j)$ allows us to embed models in which some price adjustments are free, i.e. they do not require paying the menu cost ψ, as occurs in the Calvo model, or the model with Price plans described in more details in Section D.1.

The interpretation of this problem is that the firm "tracks" the prices that maximize instantaneous profits from the n products. In this interpretation a monopolist sells n goods with additively separable demands subject to costs shocks.[3] In particular, the firm faces a system of n independent demands, with constant elasticity η for each product, random multiplicative shifts in each of the demands, and a time varying marginal (and average) cost $W\, Z_i(t)$. This is a stylized version of the problem introduced by Midrigan (2011) where the elasticity of substitution between the products sold *within* the firm is the same as the elasticity of the bundle of goods sold *across* firms. The instantaneous profit maximizing price is proportional to the marginal cost, or in logs $p_i^*(t) = \log W + \log Z_i(t) + \log(\eta/(\eta - 1))$. In this case we assume that the log of the marginal cost evolves as a random walk with drift so that $p_i^*(t)$ inherits this property. The period cost is a second order expansion of the profit function with respect to the vector of the log of prices, around the prices that

[2] Namely a sequence of shocks $\{\mathcal{W}(t)\}$ and a sequence of price changes $\{\Delta p_i(\tau_j)\}$.

[3] An alternative interpretation assumes the firm is subject to demand shocks. The demands are linear in its own price, and have zero cross partials with respect to the other prices. The marginal costs of producing each of the products are also identical and assumed to be linear. The intercepts of each of the n demands follow independent standard BMs. In this alternative interpretation the firm's profits are the sum of the n profit functions derived in the seminal work by Barro (1972), so that our ψ is his γ and our B is his θ, as defined in his equation (12).

maximize current profits (see Appendix E.2 for a detailed presentation of this interpretation). The units of the objective function are (lost) profits normalized by the maximum profits of producing *one* good. The first order price-gap terms in the expansion are zero because we are expanding around $p^*(t)$. There are no second order cross terms due to the separability of the demands. Thus we can write the problem in terms of the gap between the actual price and the profit maximizing price: $x(t) = p(t) - p^*(t)$. Under this approximation B is given by $B = (1/2)\eta(\eta - 1)$. Likewise, the fixed cost ψ is measured relative to the maximum profits of producing *one* good. Clearly all that matters to characterize the decision rules is the ratio of B to ψ, for which purpose the units in which we measure them is immaterial.

D.1 Alternative price setting technologies and decision rules

Our setup allows us to consider several modeling environments for the price setting problem. They differ in the technology that is assumed available to the firm for setting prices. Next we describe the key choices, and corresponding key parameters, that allow us to consider different price setting environments.

- First, we allow for multi product firms each producing and selling n goods (this embeds earlier papers with $n = 2$ and $n = 3$ as in Midrigan (2011); Bhattarai and Schoenle (2014), and the general model allowing for any integer n by Alvarez and Lippi (2014)). The key assumption is that once the menu cost is paid the firm can reprice all of its goods. This assumption is useful in generating the small price adjustments which are present in the data. This assumption changes the propagation mechanism by decreasing the amount of "selection" of price responses after the shock (i.e. the fact that price changes after a positive monetary shock tend to share some key features, i.e. to be mostly large and positive price increases). The baseline multiproduct model assumes that all price changes are costly, so that the indicator function $\mathcal{I}_c(\tau_j) = 1$ for all τ_j. A common feature of this setup, that extends to all environments described below, is that the menu costs gives rise to an inaction region, i.e. a region of the state space where the firm will optimally decide not to adjust prices (unless a free adjustment opportunity occurs). For symmetric problems (namely problems with a symmetric return function and no drift in the law of motion of price gaps) the inaction region is a set of points whose distance from the origin is smaller than an optimally chosen threshold $\bar{x} > 0$. In particular, the curvature of the firm's profits (B), fixed cost of adjustment (ψ), the volatility of the cost (σ), and the number of products (n), determine the optimal threshold $\bar{x}$.[4] Firms adjust the n prices the first time that norm of the deviations of their prices relative to the static markup reaches $\bar{x}$.

- Second, we allow for the menu cost to be random. In particular we introduce this feature by assuming that with some Poisson probability ζdt the menu cost is zero, so that upon such events $\mathcal{I}_c(\tau_j) = 0$. This assumption injects some free adjustments (a la Calvo) in an otherwise standard menu cost model, an assumption first explored by Nakamura and Steinsson (2010). The setup also allows to mix any amount of the

[4] One can show that the discount factor, as well as the decision of other firms have an order of magnitude smaller effect on the firm's profit, and thus can be ignored.

random-free adjustments with the multi-product assumption described above, a setting explored in Alvarez, Le Bihan, and Lippi (2016). As for the multi product assumption, these random adjustments generate small price changes and also reduce the amount of selection of price changes, by severing the link between the adjustment probability and the firm's state. Given the parameters of the firm's problem, there is still an optimally determined threshold $\bar{x}$ for the deviation of the firms prices relative to the static profit maximizing price so that firms will pay the fixed cost and change prices the first time the absolute value of the deviation reaches $\bar{x}$.[5] Thus, price changes will take place the first time that either a free adjustment opportunity occur, or that $\bar{x}$ is reached. To briefly summarize how much Calvo-type adjustments vs the regular menu-cost adjustment one model has we use the measure $\ell \equiv \zeta/N \in (0,1)$ where N denotes the expected number of price changes per period. Thus $\ell = 0$ is the canonical menu cost model with no free adjustments, and $\ell = 1$ denotes the Calvo model where all adjustments occur at exogenous random times (this obtains in our model as $\zeta > 0$ and $\psi \to \infty$). In Alvarez, Le Bihan, and Lippi (2016) we show that of the fraction ℓ is a 1-1 function of $\phi = \zeta/(\sigma^2/\bar{x}^2)$, a combination of the parameters ζ, σ^2 and the optimally determined threshold.

- Third, the model setup allows us to consider "price plans". A price plan is a set of two prices, say a high and a low price. Changing the plan entails a "menu cost", except at exponentially distributed time where it change is free –this imitates the set up of Calvo$^+$ model but it is applied to a price plan. Importantly either price within the current plan can be charged at any point in time and freely replaced by the other. While in the canonical model the set of prices to be chosen upon paying the menu cost contains only one price, the model with plans allows for some price flexibility between plan- adjustments. This model, first explored by Eichenbaum, Jaimovich, and Rebelo (2011), possibly in combination with the random-free adjustment described above, allows a large number of price changes to coexist with a few distinct price points (such as a price oscillating between a high and a low value). We study this model in detail, and obtain analytical characterization of its decision rules in Alvarez and Lippi (2018). The optimal decision rules determined both when price plan are changed, what are the prices within a plan, and how prices are changed within a plan. For symmetric problems the optimally determined prices within the plan will be symmetric about zero, namely $\pm\tilde{x}$. The optimal decision for price plans are similar than those for price changes in the Calvo+ model: they are changed the first time that the deviation of the ideal prices to the average price within the plan reaches a threshold –say $\bar{x}$– or that a free plan adjustment opportunity occurs.[6] As in the Calvo$^+$ model the size of the barrier depends on the parameter ϕ which can be mapped into ℓ, the fraction of free adjustment on *price plans*. An important element of the setup with plans is the function $\rho(\ell)$, which gives the size of the optimal price $\tilde{x}$ as a function of the optimal threshold as follows: $\tilde{x} = \rho(\ell)\bar{x}$.

[5] The determinants of $\bar{x}$ are the same as in the multiproduct model described above, with the exception on the number of products n.

[6] The determinants of $\bar{x}$ are the same as in the multiproduct and Calvo$^+$ models described above, with the exception on the number of products n.

D.2 Key properties of the economic environment

Next we summarize the features of the environment outlined above that are important to analyse the price setting problem. Each of them is detailed in Appendix E.

- First, the nominal wage is proportional to the money supply, so that monetary shocks immediately translate into a higher nominal marginal cost for firms. In particular, a once and for all change in the level of money starting at $t = 0$ corresponds to an once an for all change in the level of nominal wages starting at $t = 0$ of the same proportion.

- Second, the setup has no strategic complementarities, meaning that the price maximizing decision of each firm depends only on its own marginal cost and not on the prices of other firms. The only general equilibrium feedback on the decision of each single firm occurs through the impact on pricing decisions and level of the firm's demand is fully captured in the level of aggregate activity $c(t)$, which we argue next that can be ignored for small shocks.

- Third, the general equilibrium feedback, namely the effect of the aggregate economic activity c on the firm's decision is second order.[7] This implies that using the "steady-state" decision rules to analyze a transition to a (new) steady state after an aggregate shock provides an accurate description of the optimal firm behavior.

- Fourth, using the second and third features we can show that in order to characterize the response of the aggregate economy to a shock it is necessary to follow the firms until their *first* adjustment after the shock. This first adjustment corresponds to the first price change for the Calvo$^+$ and multiproduct models, and to the first plan change in the price plan mode. The reason for this result is that after the first adjustment the firm fully responds to the monetary shock (which has altered its nominal marginal costs), and all subsequent adjustments do *not* contribute to the aggregate prices or output, i.e. their subsequent price changes have zero expected value.

E Details on The General Equilibrium Set-Up

We next derive the GE setup that underlies the problem in equation (4).

The first order conditions for the household problem are (with respect to $L, m, c, c_{k,i}$):

$$0 = e^{-rt}\alpha - \lambda(1+\tau_L)Q(t)W(t) \tag{5}$$

$$0 = e^{-rt}\frac{1}{M(t)} - \lambda Q(t)R(t) \tag{6}$$

$$0 = e^{-rt}c(t)^{-\epsilon} - \lambda Q(t)P(t) \tag{7}$$

$$0 = e^{-rt}c(t)^{-\epsilon}c(t)^{1/\eta}c_{k,i}(t)^{-1/\eta}A_{k,i}(t)^{1/\eta} - \lambda Q(t)P_{k,i}(t) \tag{8}$$

[7] We show that these effects are, for given decision rules, of third order on the profit function, relative to the static maximizing prices.

where λ is the Lagrange multiplier of the agent budget constraint. If the money supply follows $M(t) = M(0)\exp(\mu t)$, then in an equilibrium (see Section E.1 for the derivation)

$$\lambda = \frac{1}{(\mu + r)M(0)} \quad \text{and for all } t: \; R(t) = r + \mu \;, \; W(t) = \frac{\alpha}{1+\tau_L}(r+\mu)M(t) \tag{9}$$

Moreover the foc for L and the one for c give the output equation

$$c(t)^{-\epsilon} = \frac{\alpha}{1+\tau_L}\frac{P(t)}{W(t)} \tag{10}$$

From the household's f.o.c. of $c_{k,i}(t)$ and $L(t)$ we can derive the demand for product i of firm k, given by:

$$c_{k,i}(t) = c(t)^{1-\epsilon\eta} A_{k,i}(t) \left(\frac{\alpha}{1+\tau_L} \frac{P_{k,i}(t)}{W(t)} \right)^{-\eta} \tag{11}$$

In the impulse response analysis we assume $\mu = 0$, $\tau_L = 0$, and that the initial value of $M(0)$ is such that $M(0)/P(0)$, computed using the invariant distribution of prices charged by firms, is different from its steady state value.

The nominal profit of a firm k from selling product i at price $P_{k,i}$, given the demand shock is $A_{k,i}$, marginal cost is $Z_{k,i}$, nominal wages are W and aggregate consumption c, is (we omit the time index):

$$c^{1-\epsilon\eta} A_{k,i} \left(\frac{\alpha}{1+\tau_L} \frac{P_{k,i}}{W} \right)^{-\eta} [P_{k,i} - W Z_{k,i}]$$

or, collecting $WZ_{k,i}$ and using that $A_{k,i}\, Z_{k,i}^{1-\eta} = 1$, gives

$$W c^{1-\epsilon\eta} \left(\frac{\alpha}{1+\tau_L} \frac{P_{k,i}}{W Z_{k,i}} \right)^{-\eta} \left[\frac{P_{k,i}}{W Z_{k,i}} - 1 \right]$$

so that the nominal profits of firm k from selling product i with a price gap $x_{k,i}$ is

$$W(t)\, c(t)^{1-\epsilon\eta}\, \Pi(x_{k,i}(t)) \quad \text{where} \quad \Pi(x_{k,i}) \equiv \left(\frac{\alpha}{1+\tau_L} \frac{\eta}{\eta-1} \right)^{-\eta} e^{-\eta\, x_{k,i}} \left[e^{x_{k,i}} \frac{\eta}{\eta-1} - 1 \right] \tag{12}$$

where we rewrote the actual markup in terms of the price gap $x_{k,i}$ defined as

$$x_{k,i} \equiv \log P_{k,i} - \log\left(\frac{\eta}{\eta-1} W Z_{k,i} \right) \tag{13}$$

i.e. the percent deviation of the current price from the static profit maximizing price. This shows that the price gap $x_{k,i}$ is sufficient to summarize the value of profits for product i. Note also that, by simple algebra, $\Pi(x_{k,i})/\Pi(0) = \; e^{-\eta\, x_{k,i}}\, [1 + \eta\, e^{x_{k,i}} - \eta]$, which we use below.

Next we show that the ideal price index $P(t)$, i.e. the price of one unit of the composite consumption good, can be fully characterized in terms of the price gaps. Using the definition of total expenditure (omitting time index) $P\, c = \int_0^1 \sum_{i=1}^n (P_{k,i}\; c_{k,i})\, dk$, replacing $c_{k,i}$ from equation (11), and using the first order condition with respect to c to substitute for the $c^{-\epsilon}$ term, gives

$$P = W \left(\int_0^1 \sum_{i=1}^n \left(\frac{P_{k,i}}{W Z_{k,i}} \right)^{1-\eta} dk \right)^{\frac{1}{1-\eta}} \tag{14}$$

which is the usual expression for the ideal price index, and can be written in terms of the price gaps using $\frac{P_{k,i}}{WZ_{k,i}} = e^{x_{k,i}} \frac{\eta}{\eta-1}$.

Using equation (14) to replace P/W in the aggregate output equation (10) we express the equilibrium level of output as a function of the firms' price gaps $x_{k,i}$

$$c(t) = \left(\frac{\alpha}{1+\tau_L} \frac{\eta}{\eta-1} \right)^{-\frac{1}{\epsilon}} \left(\int_0^1 \sum_{i=1}^n e^{(1-\eta)x_{k,i}} dk \right)^{\frac{1}{\epsilon(\eta-1)}} \tag{15}$$

so we can use the invariant distribution of price gaps to compute the equilibrium level of steady state output.

E.1 On the relation between Money and Wages

The above setup has a convenient property, namely that the equilibrium nominal interest rate R is constant and that nominal wages W are proportional to the money supply.

Taking the log of the first order condition for M, equation (6), gives

$$-rt - \log M(t) = \log \lambda + \log Q(t) + \log R(t)$$

or, using $M(t) = M(0)e^{\mu t}$

$$-rt - \mu t + \log M(0) = \log \lambda - \int_0^t R(s)ds + \log R(t)$$

Now differentiate with respect to time to get $-r - \mu = -R(t) + \frac{\dot{R}(t)}{R(t)}$ which shows that the constant $R(t) = r + \mu$ is a solution.

Use the foc for M and the one for labor L, equation (5) and equation (6), to get

$$W(t) = \frac{\alpha}{1+\tau_L} R(t) M(t) = \frac{\alpha}{1+\tau_L} (\mu + r) M(t)$$

where the last equality uses that the interest rate is constant. Thus in this economy this economy nominal wages are proportional to the money supply: if M_t increases by 1% the same happens to nominal wages. Notice that we can establish this fact without solving for P_t or c_t.

Taking the log of the foc for c equation (7) further shows that

$$-rt - \epsilon \log c(t) = \log \lambda - \int_0^t R(s)ds + \log P(t)$$

Now differentiate with respect to time to get $\epsilon \frac{\dot{c}(t)}{c(t)} = \mu - \frac{\dot{P}(t)}{P(t)}$ which shows the relation between real balances and consumption.

E.2 The General Equilibrium Set-Up: Firms

We assume that if firm k adjusts any of its n nominal prices at time t it must pay a fixed cost equal to ψ_L units of labor. We express these units of labor as a fraction ψ of the steady state frictionless profits from selling one of the n products, i.e. the dollar amount that has to be paid in the event of a price adjustment at t is $\psi_L W(t) = \psi\ W(t)\bar{c}^{1-\eta\epsilon}\,\Pi(0)$. To simplify notation, we omit the firm index k in what follows, and denote by p the vector of price gaps and by x_i its $i-th$ component.

The time 0 problem of a firm selling n products that starts with a price gap vector x is to choose $\{\boldsymbol{\tau}, \boldsymbol{\Delta p}\} \equiv \{\tau_j, \Delta p_i(\tau_j)\}_{j=1}^{\infty}$ to minimize the negative of the expected discounted (nominal) profits net of the menu cost. The signs are chosen so that the value function is equivalent to the loss function in equation (4):

$$-\mathbb{E}\left[\int_0^{\infty} e^{-rt}\left(\sum_{i=1}^{n} W(t)\,c(t)^{1-\epsilon\eta}\,\Pi\left(x_i(t)\right)\right)dt - \sum_{j=1}^{\infty} e^{-r\tau_j} W(t)\ \psi_L \,\middle|\, x(0)=x\right]$$

Letting $\hat{\Pi}(x_i) \equiv \Pi(x_i)/\Pi(0)$, using that equilibrium wages are constant $W(t)/\bar{W} = e^{\delta}$, and the parameterization of fixed cost in terms of steady state profits: $\psi_L = \psi\,\bar{c}^{1-\eta\epsilon}\,\Pi(0)$ gives (where bars denote steady state values):

$$\mathcal{V}(\boldsymbol{\tau}, \boldsymbol{\Delta p}, \mathbf{c}; x) \equiv -\bar{W}e^{\delta}\ \bar{c}^{1-\epsilon\eta}\Pi(0)\,\mathbb{E}\left[\int_0^{\infty} e^{-rt}\sum_{i=1}^{n}\mathcal{S}\left(c(t), x_i(t)\right)\,dt - \sum_{j=1}^{\infty} e^{-r\tau_j}\psi \;\middle|\; x(0)=x\right] \tag{16}$$

subject to equation (3), $\Delta p_i(\tau_j) \equiv \lim_{t\downarrow\tau_j} x_i(t) - \lim_{t\uparrow\tau_j} x_i(t)$ for all $i \le n$ and $j \ge 0$, where $\boldsymbol{c} = (c(t))_{t\ge 0}$ and where the function $\mathcal{S} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ gives the normalized per-product profits as a function of aggregate consumption c and the price gap of the i-th product x_i as follows:

$$\mathcal{S}(c, x_i) \equiv \left(\frac{c}{\bar{c}}\right)^{1-\eta\epsilon}\ \hat{\Pi}\left(x_i\right) \quad = \quad \left(\frac{c}{\bar{c}}\right)^{1-\eta\epsilon}\ e^{-\eta\, x_i}\left[1+\eta\, e^{x_i}-\eta\right]\ .$$

Expanding $\mathcal{S}(c, x_i)$ around $c = \bar{c}, x_i = 0$ and using that:

$$\left.\frac{\partial \mathcal{S}(c,x_i)}{\partial x_i}\right|_{x_i=0} = \left.\frac{\partial^2 \mathcal{S}(c,x_i)}{\partial x_i\,\partial c}\right|_{x_i=0} = 0\ ,\ \left.\frac{\partial^2 \mathcal{S}(c,x_i)}{\partial x_i\,\partial x_i}\right|_{x_i=0, c=\bar{c}} = \eta(1-\eta)$$

into equation (16), we obtain:

$$\mathcal{V}(\boldsymbol{\tau}, \boldsymbol{\Delta p}, \mathbf{c}; x) = \bar{W}\,\Pi(0)\,\bar{c}^{1-\eta\epsilon}\,e^{\delta}\;\Bigg\{\mathbf{V}(\boldsymbol{\tau}, \boldsymbol{\Delta p}; x) - \frac{1}{r}$$

$$- \quad (1-\epsilon\eta)\int_0^\infty e^{-rt}\left(\frac{c(t)-\bar{c}}{\bar{c}} + \frac{1}{2}\eta\epsilon\left(\frac{c(t)-\bar{c}}{\bar{c}}\right)^2 - \frac{1}{6}\eta\epsilon(1+\eta\epsilon)\left(\frac{c(t)-\bar{c}}{\bar{c}}\right)^3\right)dt$$

$$- \quad \mathbb{E}\left[\int_0^\infty e^{-rt}\frac{(2\eta-1)\eta(\eta-1)}{6}\left(\sum_{i=1}^n x_i(t)^3\right)\,dt\;\middle|\; x(0)=x\right]$$

$$+ \quad \mathbb{E}\left[\int_0^\infty e^{-rt}(1-\epsilon\eta)\frac{\eta(\eta-1)}{2}\left(\frac{c(t)-\bar{c}}{\bar{c}}\;\sum_{i=1}^n x_i(t)^2\right)\,dt\;\middle|\; x(0)=x\right]$$

$$+ \quad \mathbb{E}\left[\int_0^\infty e^{-rt}o(||(x(t),c(t)-\bar{c})||^3)\,dt\;\middle|\; x(0)=x\right]\Bigg\}$$

where $\mathbf{V}(\boldsymbol{\tau}, \boldsymbol{\Delta p}; x)$ is given by equation (4) with $B = (1/2)\eta(\eta-1)$. We can then write:

$$\mathcal{V}(\boldsymbol{\tau}, \boldsymbol{\Delta p}, \mathbf{c}; x) \;=\; \Upsilon e^{\delta}\,\mathbf{V}(\boldsymbol{\tau}, \boldsymbol{\Delta p}; x) + \mathbb{E}\left[\int_0^\infty e^{-rt}o\left(||(x(t),c(t)-\bar{c})||^2\right)\,dt\;\middle|\; x(0)=x\right] + \iota(\delta,\mathbf{c})$$

where the constant $\Upsilon = \bar{W}\,\Pi(0)\,\bar{c}^{1-\eta\epsilon}$ is the per product maximum (frictionless) nominal profits in steady state, and where the function ι does not depend of $(\boldsymbol{\tau}, \boldsymbol{\Delta p})$.

F IRF in pure Calvo Model

As $\bar{x} \to \infty$ then the spectrum (the set of eigenvalues) is no longer discrete. It is a continuum given by: $(-\infty, -\zeta]$. We derive this as a limit as $\bar{x} \to \infty$. It turns out that the limit as $\sigma^2 \to 0$ gives the same impulse response, but it has different interpretations in terms of eigenvalues-eigenfunctions.

F.1 Alternative base using sines and cosines.

To take the limit as $\bar{x} \to \infty$ it is convenient to use a base with both sines and cosines. Fix $0 < \bar{x} < \infty$ and let the eigenfunctions be:

$$\varphi_j(x) = \cos\left(x\frac{j\pi}{2\bar{x}}\right) \text{ for } j = 1, 3, 5\ldots$$
$$\varphi_j(x) = \sin\left(x\frac{j\pi}{2\bar{x}}\right) \text{ for } j = 2, 4, 6, \ldots$$

We note that they satisfy the boundary conditions $\varphi_j(\bar{x}) = \varphi_j(-\bar{x}) = 0$ for all $j = 1, 2, 3, \ldots$ and the o.d.e.:

$$(\zeta + \lambda_j)\varphi_j(x) = \varphi_j''(x)\frac{\sigma^2}{2} = -\varphi_j(x)\left(\frac{j\pi}{2\bar{x}}\right)^2 \frac{\sigma^2}{2} \text{ for } j = 1, 3, 5 \ldots$$
$$(\zeta + \lambda_j)\varphi_j(x) = \varphi_j''(x)\frac{\sigma^2}{2} = -\varphi_j(x)\left(\frac{j\pi}{2\bar{x}}\right)^2 \frac{\sigma^2}{2} \text{ for } j = 2, 4, 6, \ldots$$

and thus the eigenvalues are, as obtained before:

$$\lambda_j = -\zeta - \left(\frac{j\pi}{2\bar{x}}\right)^2 \frac{\sigma^2}{2} \text{ for } j = 1, 2, 3, \ldots$$

Thus we can write any f for which $\int_{-\bar{x}}^{\bar{x}} [f(x)]^2 dx < \infty$ as:

$$f(x) = a_0 + \sum_{j=1,3,5,\ldots} a_j[f]\varphi_j(x) + \sum_{j=2,4,6,\ldots} b_j[f]\varphi_j(x) \text{ for all } x \in [-\bar{x}, \bar{x}]$$

Note that φ_j are even (symmetric) functions for $j = 1, 3, 5, \ldots$ and φ_j are odd (symmetric) functions for $j = 2, 4, 6, \ldots$. The coefficients $b[f]$ and $a[f]$ are given by:

$$a_j[f] = \frac{1}{\bar{x}} \int_{-\bar{x}}^{\bar{x}} \cos\left(x\frac{j\pi}{2\bar{x}}\right) f(x)dx \text{ for } j = 1, 3, 5 \ldots$$
$$b_j[f] = \frac{1}{\bar{x}} \int_{-\bar{x}}^{\bar{x}} \sin\left(x\frac{j\pi}{2\bar{x}}\right) f(x)dx \text{ for } j = 2, 4, 6, \ldots$$

Since we are interested in function $f(x)$ –and $p(x, 0)$– that are odd (antisymmetric) then $a_j[f] = 0$. Thus we can write $f(x)$ for all $x \in [-\bar{x}, \bar{x}]$ as:

$$\begin{aligned} f(x) &= \sum_{j=2,4,6,\ldots} b_j[f] \sin\left(x\frac{j\pi}{2\bar{x}}\right) = \sum_{j=2,4,6,\ldots} \left[\frac{1}{\bar{x}} \int_{-\bar{x}}^{\bar{x}} \sin\left(\hat{x}\frac{j\pi}{2\bar{x}}\right) f(\hat{x})d\hat{x}\right] \sin\left(x\frac{j\pi}{2\bar{x}}\right) \\ &= \sum_{j=1}^{\infty} \left[\frac{1}{\bar{x}} \int_{-\bar{x}}^{\bar{x}} \sin\left(\hat{x}\frac{j\pi}{\bar{x}}\right) f(\hat{x})d\hat{x}\right] \sin\left(x\frac{j\pi}{\bar{x}}\right) \\ &= \sum_{j=1}^{\infty} \frac{1}{\pi} \left[\int_{-\bar{x}}^{\bar{x}} \sin\left(\hat{x}\frac{j\pi}{\bar{x}}\right) f(\hat{x})d\hat{x}\right] \sin\left(x\frac{j\pi}{\bar{x}}\right) \frac{\pi}{\bar{x}} \end{aligned}$$

F.2 Limiting case as $\bar{x} \to \infty$.

We can let $\omega = (j\pi)/\bar{x}$ and $d\omega = \pi/\bar{x}$ we have a Riemman integral, we have:

$$
\begin{aligned}
f(x) &= \lim_{\bar{x}\to\infty} \sum_{j=1}^{\infty} \frac{1}{\pi} \left[\int_{-\bar{x}}^{\bar{x}} \sin\left(\hat{x}\frac{j\pi}{\bar{x}}\right) f(\hat{x}) d\hat{x} \right] \sin\left(x\frac{j\pi}{\bar{x}}\right) \frac{\pi}{\bar{x}} \\
&= \frac{1}{\pi} \int_0^{\infty} \left[\int_{-\infty}^{\infty} \sin\left(\hat{x}\omega\right) f(\hat{x}) d\hat{x} \right] \sin\left(x\omega\right) d\omega \quad \text{for all } x \in (-\infty, \infty)
\end{aligned}
$$

We require the integral inside the square brackets to be well defined. A sufficient condition for that is that the function f be integrable, i.e. $\int_{-\infty}^{\infty} |f(x)| dx < \infty$. This is the identity for the Fourier transform of an odd function f, which is the pure imaginary part of the Fourier transform.

The eigenvalues and eigenfunctions are indexed by $\omega \geq 0$ as:

$$
\lambda(\omega) = -\zeta - \omega^2\sigma^2/2 \text{ and } \varphi(x,\omega) = \sin(x\omega) \text{ for all } x \in \mathbb{R} \text{ and all } \omega \in \mathbb{R}_+ .
$$

In this case we have

$$
\begin{aligned}
Y(t) &= \int_{-\infty}^{\infty} E\left[f(x(t))|x(0) = x\right] p(x,0) dx \\
&= \int_{-\infty}^{\infty} E\left[\frac{1}{\pi} \int_0^{\infty} \left[\int_{-\infty}^{\infty} \sin\left(\hat{x}\omega\right) f(\hat{x}) d\hat{x} \right] \sin\left(x(t)\omega\right) d\omega \Big| x(0) = x \right] p(x,0) dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\pi} \int_0^{\infty} \left[\int_{-\infty}^{\infty} \sin\left(\hat{x}\omega\right) f(\hat{x}) d\hat{x} \right] E\left[\sin\left(x(t)\omega\right) d\omega \Big| x(0) = x \right] p(x,0) dx \\
&= \int_{-\infty}^{\infty} \left(\frac{1}{\pi} \int_0^{\infty} e^{\lambda(\omega)t} \left[\int_{-\infty}^{\infty} \sin\left(\hat{x}\omega\right) f(\hat{x}) d\hat{x} \right] \sin\left(x\omega\right) d\omega \right) p(x,0) dx \\
&= \frac{1}{\pi} \int_0^{\infty} e^{\lambda(\omega)t} \left[\int_{-\infty}^{\infty} \sin\left(\hat{x}\omega\right) f(\hat{x}) d\hat{x} \right] \left[\int_{-\infty}^{\infty} \sin\left(x\omega\right) p(x,0) dx \right] d\omega
\end{aligned}
$$

The problem to properly defined the IRF $Y(t)$ in the pure Calvo case when $\bar{x} \to \infty$ case is that the function $f = -x$ is not integrable, and hence its Fourier transform in not well defined. The eigenvalue-eigenfunctions converge to a well defined continuum. Indeed the first and second eigenvalue becomes arbitrary close to each other, and both converge to ζ.

G Random Menu cost aka Calvo$^+$ model

For completeness the Kolmogorov forward equation indicates that the time derivative of p is:

$$
p_t(x,t) = \frac{\sigma^2}{2} p''(x,t) - \zeta p(x,t) \text{ for all } x \in [\underline{x}, \bar{x}] \text{ and } t > 0
$$

with boundary conditions $p(\underline{x}, t) = p(\bar{x}, t) = 0$ for all $t \rangle 0$, since these are exit points. Thus we can define the operator $\mathcal{B}(p)$, the forward Kolmogorov equation as:

$$\mathcal{B}(p)(x) = \frac{\sigma^2}{2} p''(x) - \zeta p(x) \text{ for all } x \in [\underline{x}, \bar{x}]$$

for any function $p \geq 0$ where $p(\underline{x}) = p(\bar{x})$ and for which $\int_{\underline{x}}^{\bar{x}} p(x)^2 dx < \infty$.

We define the inner product as:

$$\langle f, p \rangle = \int_{\underline{x}}^{\bar{x}} f(x) p(x) dx$$

The linear operator $\mathcal{B}$ is the adjoint of $\mathcal{A}$ if

$$\langle \mathcal{A}(f), p \rangle = \langle f, \mathcal{B}(p) \rangle$$

or

$$\int_{\underline{x}}^{\bar{x}} \left[\frac{\sigma^2}{2} f''(x) - \zeta f(x) \right] p(x) dx = \int_{\underline{x}}^{\bar{x}} f(x) \left[\frac{\sigma^2}{2} p''(x) - \zeta p(x) \right] dx$$

This holds if and only if

$$\int_{\underline{x}}^{\bar{x}} \left[\frac{\sigma^2}{2} f''(x) \right] p(x) dx = \int_{\underline{x}}^{\bar{x}} \left[\frac{\sigma^2}{2} p''(x) \right] f(x) dx$$

and using integration by parts gives:

$$\int_{\underline{x}}^{\bar{x}} \frac{\sigma^2}{2} f''(x) p(x) dx = \frac{\sigma^2}{2} f'(x) p(x) |_{\underline{x}}^{\bar{x}} - \int_{\underline{x}}^{\bar{x}} \frac{\sigma^2}{2} f'(x) p'(x) dx$$

and integrating again:

$$\int_{\underline{x}}^{\bar{x}} \frac{\sigma^2}{2} f''(x) p(x) dx = \frac{\sigma^2}{2} f'(x) p(x) |_{\underline{x}}^{\bar{x}} - \frac{\sigma^2}{2} f(x) p'(x) |_{\underline{x}}^{\bar{x}} \int_{\underline{x}}^{\bar{x}} \frac{\sigma^2}{2} f(x) p''(x) dx$$

Thus we require:

$$0 = f'(\bar{x}) p(\bar{x}) - f'(\underline{x}) p(\underline{x}) - f(\bar{x}) p'(\bar{x}) + f(\underline{x}) p'(\underline{x})$$

Note that given $p(\bar{x}) = p(\underline{x}) = 0$ it implies $f(\bar{x}) = f(\underline{x}) = 0$.

The operator $\mathcal{B}(p)$ is also related to the the discrete time $B(p)$ as follows:

$$\mathcal{B}(p)(x) = \lim_{\Delta \downarrow 0} \frac{p_{t+\Delta}(x) - p_t(x)}{\Delta}.$$

H Discrete change of volatility

We want to compute the short run impulse response of a Golosov-Lucas standard menu cost economy that moves from a low idiosyncratic variance σ_0 to a permanently higher variance σ_1. Immediately after this change, there is a monetary shock δ.

To compute the corresponding "short term" impulse response we need the coefficients for the Fourier series for the derivative of the initial distribution $\bar{p}'(x;\bar{x}_0)$ when $\bar{x}_0 < \bar{x}_1$, where $\bar{x}_i$ is the threshold that correspond to σ_i. The base are $\varphi_j(x) = \sin\left(\frac{x+\bar{x}_1}{2\bar{x}_1} j\pi\right)$ defined in $x \in [-\bar{x}_1, \bar{x}_1]$. The function of interest is:

$$p(x,0) = \begin{cases} 0 & \text{if } x \in [-\bar{x}_1, -\bar{x}_0] \\ \bar{p}'(x;\bar{x}_0) = +\frac{1}{\bar{x}_0^2} & \text{if } x \in [-\bar{x}_0, 0) \\ \bar{p}'(x;\bar{x}_0) = -\frac{1}{\bar{x}_0^2} & \text{if } x \in (0, \bar{x}_0] \\ 0 & \text{if } x \in [\bar{x}_0, \bar{x}_1] \end{cases}$$

We have:

$$\begin{aligned} b_j[p(x,0)] &= \frac{1}{\bar{x}_1}\int_{-\bar{x}_0}^{0} \frac{1}{\bar{x}_0^2} \sin\left(\frac{x+\bar{x}_1}{2\bar{x}_1} j\pi\right) dx + \frac{1}{\bar{x}_1}\int_{0}^{\bar{x}_0} \frac{1}{-\bar{x}_0^2} \sin\left(\frac{x+\bar{x}_1}{2\bar{x}_1} j\pi\right) dx \\ &= \frac{2\bar{x}_1}{\bar{x}_1\bar{x}_0^2}\left[\frac{\cos\left(\frac{(\bar{x}_1-\bar{x}_0)}{2\bar{x}_1}(j\pi)\right) - \cos\left(\frac{\pi j}{2}\right)}{\pi j}\right] - \frac{2\bar{x}_1}{\bar{x}_1\bar{x}_0^2}\left[\frac{\cos\left(\frac{\pi j}{2}\right) - \cos\left(\frac{(\bar{x}_1+\bar{x}_0)}{2\bar{x}_1}(j\pi)\right)}{\pi j}\right] \\ &= -\frac{4\bar{x}_1}{\bar{x}_1\bar{x}_0^2\pi j}\cos\left(\frac{\pi j}{2}\right) + \frac{2\bar{x}_1}{\bar{x}_1\bar{x}_0^2\pi j}\left[\cos\left(\frac{(\bar{x}_1-\bar{x}_0)}{2\bar{x}_1}(j\pi)\right) + \cos\left(\frac{(\bar{x}_1+\bar{x}_0)}{2\bar{x}_1}(j\pi)\right)\right] \end{aligned}$$

which equals zero for j odd.[8]

We want to consider $\bar{x}_0 = \left(6\frac{\psi}{B}\sigma_0^2\right)^{\frac{1}{4}} < \bar{x}_1 = \left(6\frac{\psi}{B}\sigma_1^2\right)^{\frac{1}{4}}$ with $\sigma_0 < \sigma_1$. Recall that $b_j[f] = b_j[-x] = 4\bar{x}_1/(j\pi)$ for j even and 0 otherwise, so that the only coefficient that will enter the IRF will be the even ones.

We let $N_i = (\sigma_i/\bar{x}_i)^2$ be the average number of price changes in the steady state corre-

[8] Notice that for $\bar{x} = \bar{x}_0$ we have: $b_j[p(x,0)] = \frac{2}{\bar{x}_1^2\pi j}\left[-2\cos\left(\frac{\pi j}{2}\right) + \cos(0) + \cos(j\pi)\right]$. For $j = 2, 4, 6, 8, 10\cdots$, we have:

$$-2\cos\left(\frac{\pi j}{2}\right) + \cos(0) + \cos(j\pi) = [4\,0\,4\,0\,4\,0\cdots]$$

or equal to $8/(\bar{x}_1^2\pi j)$ every four j, which is the same expression we got in the benchmark case of Section ??.

sponding to $i = 0, 1$. Thus we have:

$$Y_{SR}(t;\sigma_1,\sigma_0) = \bar{x}_1 \sum_{j=1}^{\infty} b_j[f] b_j[p(\cdot,0)] e^{-N_1 \frac{\pi^2}{8}(2j)^2 t}$$
$$= \bar{x}_1 \sum_{j=1}^{\infty} \frac{4\bar{x}}{(2j\pi)^2} \left(-\frac{4}{\bar{x}_0^2} \cos\left(\frac{\pi 2j}{2}\right) + \frac{2}{\bar{x}_0^2} \left[\cos\left(\frac{(\bar{x}_1 - \bar{x}_0)}{2\bar{x}_1} (2j\pi) \right) + \cos\left(\frac{(\bar{x}_1 + \bar{x}_0)}{2\bar{x}_1} (2j\pi) \right) \right] \right) e^{-N_1 \frac{\pi^2}{8}(2j)^2 t}$$
$$= \left(\frac{\bar{x}}{\bar{x}_0}\right)^2 \sum_{j=1}^{\infty} \frac{8}{(2j\pi)^2} \left(-2\cos(\pi j) + \left[\cos\left(\frac{(\bar{x}_1 - \bar{x}_0)}{\bar{x}_1} (j\pi) \right) + \cos\left(\frac{(\bar{x}_1 + \bar{x}_0)}{\bar{x}_1} (j\pi) \right) \right] \right) e^{-N_1 \frac{\pi^2}{8}(2j)^2 t}$$

Note that:

$$\frac{\bar{x}}{\bar{x}_0} = \left(\frac{\sigma_1}{\sigma_0}\right)^{\frac{2}{4}} = \left(\frac{\sigma_1}{\sigma_0}\right)^{\frac{1}{2}}$$

Thus

$$Y_{SR}(t;\sigma_1,\sigma_0)$$
$$= \frac{\sigma_1}{\sigma_0} \sum_{j=1}^{\infty} \frac{8}{(2j\pi)^2} \left(-2\cos(\pi j) + \left[\cos\left(\frac{(\bar{x}_1 - \bar{x}_0)}{\bar{x}_1} (j\pi) \right) + \cos\left(\frac{(\bar{x}_1 + \bar{x}_0)}{\bar{x} v} (j\pi) \right) \right] \right) e^{-N_1 \frac{\pi^2}{8}(2j)^2 t}$$
$$= \frac{\sigma_1}{\sigma_0} \sum_{j=1}^{\infty} \frac{8}{(2j\pi)^2} \left(-2\cos(\pi j) + \left[\cos\left(\frac{(\bar{x}_1 - \bar{x}_0)}{\bar{x}_1} (j\pi) \right) + \cos\left(\frac{(\bar{x}_1 + \bar{x}_0)}{\bar{x}_1} (j\pi) \right) \right] \right) e^{-N_1 \frac{\pi^2}{8}(2j)^2 t}$$
$$= \frac{\sigma_1}{\sigma_0} \sum_{j=1}^{\infty} \frac{8}{(2j\pi)^2} \left(-2\cos(\pi j) + \left[\cos\left(\left(1 - \left(\frac{\sigma_0}{\sigma_1}\right)^{1/2}\right) (j\pi) \right) + \cos\left(\left(1 + \left(\frac{\sigma_0}{\sigma_1}\right)^{1/2}\right) (j\pi) \right) \right] \right) e^{-N_1 \frac{\pi^2}{8}(2j)^2 t}$$

Note that we can write the level and derivative of the impulse response with no change in σ as:

$$Y_{SR}(t;\sigma_0,\sigma_0) = \sum_{j=1}^{\infty} \frac{8}{(2j\pi)^2} \left(-2\cos(\pi j) + 1 + \cos(2(j\pi))\right) e^{-N_0 \frac{\pi^2}{8}(2j)^2 t}$$
$$\frac{\partial Y_{SR}(t;\sigma_0,\sigma_0)}{\partial t} = \sum_{j=1}^{\infty} \frac{8}{(2j\pi)^2} \left(-2\cos(\pi j) + 1 + \cos(2(j\pi))\right) \left[-N_0 \frac{\pi^2}{8} (2j)^2\right] e^{-N_0 \frac{\pi^2}{8}(2j)^2 t}$$

We also note that

$$\sigma_1 \frac{\partial Y_{SR}(t;\sigma_0,\sigma_1)}{\partial \sigma_1}\Big|_{\sigma_1=\sigma_0}$$
$$= Y_{SR}(t;\sigma_0,\sigma_0) + \sigma_1 \sum_{j=1}^{\infty} \frac{8}{(2j\pi)^2} \left(-\sin(0) \frac{1}{2\sigma_1} + \sin(2j\pi) \frac{1}{2\sigma_1} \right) e^{-N_1 \frac{\pi^2}{8}(2j)^2 t}$$
$$+ \frac{\sigma_1}{N_1} \frac{\partial N_1}{\partial \sigma_1} \sum_{j=1}^{\infty} \frac{8}{(2j\pi)^2} \left(-N_1 \frac{\pi^2}{8} (2j)^2 \right) t \left[-2\cos(\pi j) + 1 + \cos(2j\pi)\right] e^{-N_1 \frac{\pi^2}{8}(2j)^2 t}$$

Using that $\frac{\sigma_1}{N_1}\frac{\partial N_1}{\partial \sigma_1} = 1$, and that $\sin(j\pi) = 0$ for integer j, then

$$\sigma_1 \frac{\partial Y_{SR}(t;\sigma_0,\sigma_1)}{\partial \sigma_1}|_{\sigma_1=\sigma_0}$$
$$= Y_{SR}(t) + \sum_{j=1}^{\infty} \frac{8}{(2j\pi)^2}\left(-N_1 \frac{\pi^2}{8}(2j)^2\right) t\left[-2\cos(\pi j) + 1 + \cos(2j\pi)\right] e^{-N_1 \frac{\pi^2}{8}(2j)^2 t}$$

Using the calculation above we establish the following properties:

$$\sigma_1 \frac{\partial Y_{SR}(t;\sigma_0,\sigma_1)}{\partial \sigma_1}|_{\sigma_1=\sigma_0} = Y_{SR}(t;\sigma_0,\sigma_0) + t\,\frac{\partial Y_{SR}(t;\sigma_0,\sigma_0)}{\partial t} \text{ for all } t>0 \text{ and} \tag{17}$$
$$\int_0^{\infty} \sigma_1 \frac{\partial Y_{SR}(t;\sigma_0,\sigma_1)}{\partial \sigma_1}|_{\sigma_1=\sigma_0} dt = 0 \tag{18}$$

where the second equality follows using integration by parts. We note that the following function satisfies both of these properties:

$$Y_{SR}(t;\sigma_0,\sigma_1) = Y\left(t\left(\frac{\sigma_1}{\sigma_0}\right);\sigma_0\right)\left(\frac{\sigma_1}{\sigma_0}\right) + o\left(\frac{\sigma_1}{\sigma_0}\right)$$

where $Y(t;\sigma)$ is the impulse response of an economy with volatility σ.

I Kolmogorov Forward and Kolmogorov Backward equivalence

First we state an equivalence result between the Kolmogorov Forward and Kolmogorov Backward equations.

PROPOSITION 1. Assume that Forward and Backward operators $\mathcal{H}^*$ and $\mathcal{H}$ solve the p.d.e. and boundary conditions stated above. Then, for any functions p and f, times $0 < \tau < T$ and a number Δ we have:

$$H(T+\Delta,f,p) = \int_{\underline{x}}^{\bar{x}} \mathcal{H}(f)(x,T-\tau+\Delta)\mathcal{H}^*(p)(x,\tau)dx = \int_{\underline{x}}^{\bar{x}} \mathcal{H}(f)(x,T-\tau)\mathcal{H}^*(p)(x,\tau+\Delta)dx$$

Proof. (of Proposition 1) We verify the equality for $H(T+\Delta,f,p)$ by differentiating w.r.t. Δ and evaluating at $\Delta = 0$:

$$\int_{\underline{x}}^{\bar{x}} \mathcal{H}_t(f)(x,T-\tau)\mathcal{H}^*(p)(x,\tau)dx = \int_{\underline{x}}^{\bar{x}} \mathcal{H}(f)(x,T-\tau)\mathcal{H}_t^*(p)(x,\tau)dx$$

Splitting the integral into $[\underline{x},x^*]$ and $[x^*,\bar{x}]$, and replacing the p.d.e. and cancelling $\sigma^2/2$

from both sides:

$$\int_{\underline{x}}^{x^*} \mathcal{H}_{xx}(f)(x,T-\tau)\mathcal{H}^*(p)(x,\tau)dx + \int_{x^*}^{\bar{x}} \mathcal{H}_{xx}(f)(x,T-\tau)\mathcal{H}^*(p)(x,\tau)dx$$
$$= \int_{\underline{x}}^{x^*} \mathcal{H}(f)(x,T-\tau)\mathcal{H}^*_{xx}(p)(x,\tau)dx + \int_{x^*}^{\bar{x}} \mathcal{H}(f)(x,T-\tau)\mathcal{H}^*_{xx}(p)(x,\tau)dx$$

Twice Integrating by parts the expression after the equality we have:

$$\int_{\underline{x}}^{x^*} \mathcal{H}_{xx}(f)(x,T-\tau)\mathcal{H}^*(p)(x,\tau)dx + \int_{x^*}^{\bar{x}} \mathcal{H}_{xx}(f)(x,T-\tau)\mathcal{H}^*(p)(x,\tau)dx$$
$$= \int_{\underline{x}}^{x^*} \mathcal{H}_{xx}(f)(x,T-\tau)\mathcal{H}^*(p)(x,\tau)dx + \int_{x^*}^{\bar{x}} \mathcal{H}_{xx}(f)(x,T-\tau)\mathcal{H}^*(p)(x,\tau)dx$$
$$+ \mathcal{H}(f)(x,T-\tau)\mathcal{H}^*_x(p)(x,\tau)\Big|_{\underline{x}}^{x^*} + \mathcal{H}(f)(x,T-\tau)\mathcal{H}^*_x(p)(x,\tau)\Big|_{\underline{x}}^{x^*}$$
$$- \mathcal{H}_x(f)(x,T-\tau)\mathcal{H}^*(p)(x,\tau)\Big|_{\underline{x}}^{x^*} - \mathcal{H}_x(f)(x,T-\tau)\mathcal{H}^*(p)(x,\tau)\Big|_{\underline{x}}^{x^*}$$

Cancelling the common terms from the integrals we have:

$$0 = \mathcal{H}(f)(x,T-\tau)\mathcal{H}^*_x(p)(x,\tau)\Big|_{\underline{x}}^{x^*} + \mathcal{H}(f)(x,T-\tau)\mathcal{H}^*_x(p)(x,\tau)\Big|_{\underline{x}}^{x^*}$$
$$- \mathcal{H}_x(f)(x,T-\tau)\mathcal{H}^*(p)(x,\tau)\Big|_{\underline{x}}^{x^*} - \mathcal{H}_x(f)(x,T-\tau)\mathcal{H}^*(p)(x,\tau)\Big|_{\underline{x}}^{x^*}$$

Using that $\mathcal{H}^*(p) = 0$ at the exit points, as well as the continuity of $\mathcal{H}_x(f)(x,T-\tau)$ and $\mathcal{H}^*(p)(x,\tau)$ at $x = x^*$

$$0 = \mathcal{H}(f)(x,T-\tau)\mathcal{H}^*_x(p)(x,\tau)\Big|_{\underline{x}}^{x^*} + \mathcal{H}(f)(x,T-\tau)\mathcal{H}^*_x(p)(x,\tau)\Big|_{\underline{x}}^{x^*}$$

Using the boundary conditions at $\bar{x}, x^*$ and $\underline{x}$ for $\mathcal{H}(f)(x,T-t)$:

$$0 = \mathcal{H}(f)(x^*,T-\tau)\left[\mathcal{H}^*_x(p)(x,\tau)\Big|_{\underline{x}}^{x^*} + \mathcal{H}^*_x(p)(x,\tau)\Big|_{\underline{x}}^{x^*}\right] = \mathcal{H}(f)(x^*,T-\tau)\left[\int_{\underline{x}}^{\bar{x}} \mathcal{H}^*_{xx}(p)(x,\tau)dx\right]$$
$$= \mathcal{H}(f)(x^*,T-\tau)\frac{2}{\sigma^2}\left[\int_{\underline{x}}^{\bar{x}} \mathcal{H}^*_t(p)(x,\tau)dx\right] = \mathcal{H}(f)(x^*,T-\tau)\frac{2}{\sigma^2}\left[\partial_t \int_{\underline{x}}^{\bar{x}} \mathcal{H}^*(p)(x,\tau)dx\right]$$

Using that $\mathcal{H}^*(p)$ is measure preserving, i.e. $\partial_t \int_{\underline{x}}^{\bar{x}} \mathcal{H}^*(p)(x,\tau)dx = 0$, the term in square brackets is zero, and thus we verify the equality. $\square$

Let $\alpha = \frac{x^*-\underline{x}}{\bar{x}-\underline{x}} \in (0, 1/2]$ index the asymmetry of the problem, with $\alpha = 1/2$ representing the benchmark symmetric problem where the return point is in the middle of the inaction range. The Kolmogorov backward equation defines $\mathcal{H}(f) : [\underline{x}, \bar{x}] \times \mathbb{R}_+$ a differentiable function for which:

$$\partial_t \mathcal{H}(f)(x,t) = \frac{\sigma^2}{2}\partial_{xx}\mathcal{H}(f)(x,t) - \zeta\mathcal{H}(f)(x,t) \text{ for all } x \in [\underline{x}, \bar{x}] \text{ and all } t > 0 \qquad (19)$$

with boundary conditions:

$$\mathcal{H}(f)(\underline{x},t) = \mathcal{H}(f)(x^*,t) = \mathcal{H}(f)(\bar{x},t) \tag{20}$$
$$\mathcal{H}(f)(x,0) = f(x) \text{ for all } x \in [\underline{x},\bar{x}] \tag{21}$$

The p.d.e. in equation (19) is standard. The boundary conditions in equation (20) are an implication of the fact that when the boundaries $\{\underline{x},\bar{x}\}$ are reached, the process returns to x^*.

We claim that the eigenvalues of the Kolmogorov backward operator are the same as the ones for the derived above for the forward operator. Moreover, letting η be the eigenfunctions of the Backward Kolmogorov equation, they can be constructed by a phase shift of the eigenfunctions of forward one. The next proposition verifies that indeed we have the correct p.d.e and boundary conditions:

Proposition 2. Let $\alpha = \frac{x^*-\underline{x}}{\bar{x}-\underline{x}} \in (0,1/2]$ and assume α is not rational. Let $\{\lambda_j^k\}_{j=1}^{\infty}$ for $k \in \{m,l,h\}$ be the (negative) eigenvalues for $\mathcal{H}^*$ described in Proposition ??. Then $\{\lambda_j^k\}_{j=1}^{\infty}$ for $k \in \{m,l,h\}$ are eigenvalues for $\mathcal{H}$ with the following corresponding eigenfunctions:

$$\eta_j^m(x) = \cos\left(2\pi j\left(\frac{\alpha}{2} - \frac{x-\underline{x}}{\bar{x}-\underline{x}}\right)\right)$$
$$\eta_j^l(x) = \cos\left(2\pi j\left(\frac{1}{2\alpha} - \frac{x-\underline{x}}{x^*-\underline{x}}\right)\right)$$
$$\eta_j^h(x) = \cos\left(2\pi j\left(\frac{\alpha}{2(1-\alpha)} + \frac{x-x^*}{\bar{x}-x^*}\right)\right)$$

for all $j = 1, 2, ...$ and all $x \in [\underline{x},\bar{x}]$.

Straightforward analysis reveals that the eigenfunctions $\eta_j^l(x)$ and $\eta_j^h(x)$ are symmetric around the midpoint $(\underline{x}+\bar{x})/2$. Notice that we can rewrite the eigenfunctions $\eta_j^m(x)$ as sum of a symmetric component $\eta_j^{m,s}(x)$ and an antisymmetric one $\eta_j^{m,a}(x)$ using standard trigonometric identities, so that $\eta_j^m(x) = \eta_j^{m,a}(x) + \eta_j^{m,s}(x)$. In particular we have:

$$\eta_j^{m,a}(x) \equiv \cos\left(s_j^m\right)\sin\left(\frac{x-\underline{x}}{\bar{x}-\underline{x}}2\pi j\right) \quad , \quad \eta_j^{m,s}(x) \equiv \sin\left(s_j^m\right)\cos\left(\frac{x-\underline{x}}{\bar{x}-\underline{x}}2\pi j\right) \tag{22}$$

where it appears that the eigenfunctions $\eta_j^{m,a}(x)$ are the same odd functions, up to a scaling factor, of the odd ones found for the forward equation.[9]

Proof. (of Proposition 2) The eigenfunctions and eigenvalues of $\mathcal{H}$ we look for are functions of the form

$$\mathcal{H}(\eta)(x,t) = e^{\lambda t}\eta(x)$$

[9]Notice that these eigenfunctions are not normalized ($\int_{\underline{x}}^{\bar{x}}(\eta_j^k(x))^2dx \neq 1$ for all j and $k = l,h,m$). This needs to be taken into account when computing the projections.

so that its p.d.e. and boundary conditions becomes:

$$\lambda\eta(x)=\frac{\sigma^2}{2}\eta''(x) \text{ for all } x\in[\underline{x},\bar{x}] \text{ and } \eta(\bar{x})=\eta(x^*)=\eta(\underline{x}) \tag{23}$$

This o.d.e. is solved by φ_j^k and λ_j^k. Thus we only need to verify that the boundary conditions are met.

We start with the case of λ_j^m,φ_j^m with $\varphi_j^m(x)=A\sin\left(\frac{x-\underline{x}}{\bar{x}-\underline{x}}2\pi j\right)$ for some constant A which we can normalize to one. This eigenfunction satisfies:

$$\varphi_j^m(\underline{x})=\varphi_j^m(\bar{x})=0$$

Thus, for any number s_j^m, given the definition above for η_j^m we have:

$$\eta_j^m(\bar{x})=\eta_j^m(\underline{x})$$

since $\sin(\cdot)$ has periodicity 2π. The final step is to find the value of s_j^m so that:

$$\eta_j^m(\underline{x})=\eta_j^m(x^*)$$

or

$$\sin\left(s_j^m\right)=\sin\left(\frac{x^*-\underline{x}}{\bar{x}-\underline{x}}2\pi j+s_j^m\right)$$

Using that $\sin(A+B)=\sin(A)\cos(B)+\sin(B)\cos(A)$ we have:

$$\sin\left(s_j^m\right)=\sin\left(\frac{x^*-\underline{x}}{\bar{x}-\underline{x}}2\pi j\right)\cos\left(s_j^m\right)+\sin\left(s_j^m\right)\cos\left(\frac{x^*-\underline{x}}{\bar{x}-\underline{x}}2\pi j\right)$$

or

$$\tan\left(s_j^m\right)\equiv\frac{\sin\left(s_j^m\right)}{\cos\left(s_j^m\right)}=\frac{\sin\left(\frac{x^*-\underline{x}}{\bar{x}-\underline{x}}2\pi j\right)}{1-\cos\left(\frac{x^*-\underline{x}}{\bar{x}-\underline{x}}2\pi j\right)}$$

or

$$\begin{aligned}\cot\left(s_j^m\right)&=\frac{1}{\tan\left(s_j^m\right)}=\frac{1-\cos\left(\frac{x^*-\underline{x}}{\bar{x}-\underline{x}}2\pi j\right)}{\sin\left(\frac{x^*-\underline{x}}{\bar{x}-\underline{x}}2\pi j\right)}=\frac{1}{\sin\left(\frac{x^*-\underline{x}}{\bar{x}-\underline{x}}2\pi j\right)}-\frac{\cos\left(\frac{x^*-\underline{x}}{\bar{x}-\underline{x}}2\pi j\right)}{\sin\left(\frac{x^*-\underline{x}}{\bar{x}-\underline{x}}2\pi j\right)}\\&=\csc\left(\frac{x^*-\underline{x}}{\bar{x}-\underline{x}}2\pi j\right)-\cot\left(\frac{x^*-\underline{x}}{\bar{x}-\underline{x}}2\pi j\right)\end{aligned}$$

or

$$s_j^m=\cot^{-1}\left(\csc\left(\frac{x^*-\underline{x}}{\bar{x}-\underline{x}}2\pi j\right)-\cot\left(\frac{x^*-\underline{x}}{\bar{x}-\underline{x}}2\pi j\right)\right)$$

Next we consider the cases for the other types of eigenvalues and eigenfunctions. Since they are similar, we skip part of the arguments. Using the periodicity of sin and the boundary for the φ_j^l and φ_j^h

$$\eta_j^l(x^*) = \eta_j^l(\underline{x}) \text{ and } \eta_j^h(\bar{x}) = \eta_j^h(x^*)$$

for any choice for s_j^l and s_j^h.

To solve for s_j^l we impose:

$$\eta_j^l(x^*) = \eta_j^l(\bar{x}) \implies \sin(s_j^l) = \sin\left(\frac{\bar{x}-\underline{x}}{x^*-\underline{x}}2\pi j + s_j^l\right)$$

which implies

$$s_j^l = \cot^{-1}\left(\csc\left(\frac{\bar{x}-\underline{x}}{x^*-\underline{x}}2\pi j\right) - \cot\left(\frac{\bar{x}-\underline{x}}{x^*-\underline{x}}2\pi j\right)\right)$$

Likewise to solve for s_j^h we impose:

$$\eta_j^h(x^*) = \eta_j^l(\underline{x}) \implies \sin(s_j^h) = \sin\left(\frac{\underline{x}-x^*}{\bar{x}-x^*}2\pi j + s_j^h\right)$$

which implies

$$s_j^h = \cot^{-1}\left(\csc\left(\frac{\underline{x}-x^*}{\bar{x}-x^*}2\pi j\right) - \cot\left(\frac{\underline{x}-x^*}{\bar{x}-x^*}2\pi j\right)\right)$$

J Asymmetric problem: Approximate analytic characterization.

An approximate analytic solution to finding the a_j^k coefficients can be obtained by constructing a set of orthogonalized eigenfunctions. To retain tractability we will assume that the new set of eigenfunctions is simply made by the set of odd eigenfunctions φ_j^m with $j = 1, 2, ...$ (which are orthogonal to each other) and by the symmetric eigenfunction φ_1^h, namely the one associated to the second largest eigenvalue (for concreteness we assume that $\alpha < 1/2$ so that $\bar{x} - x^* > x^* - \underline{x}$, see equation (**??**)). This solution is approximate since it does not use all the eigenfunctions.

Using the Grahm-Schmidt algorithm to orthogonalize φ_1^h with respect to the set φ_j^m gives a new orthogonal eigenfunction

$$v_1^h(x) = \varphi_1^h(x) - \sum_{j=1}^{\infty} \gamma_{1,j}^m \varphi_j^m(x) \quad \text{where} \quad \gamma_{1,j}^m = \int_{\underline{x}}^{\bar{x}} \varphi_1^h(x)\varphi_j^m(x)\,dx$$

where simple calculus gives

$$\gamma_{1,j}^m = \frac{\sqrt{(1-\alpha)}\sin(2\pi j\alpha)}{\pi\left(1-(1-\alpha)^2 j^2\right)}$$

It is also straightforward to see that since the eigenfunctions form a basis for odd functions, the function $v_1^h(x)$ that results from the orthogonalization is the symmetric component of $\varphi_1^h(x)$.[10] To simplify notation, and without loss of generality we normalize the domain of x

[10]Obviously any function $f(x)$ can be written as the sum of an even and odd component (superscript s

such that its midpoint equals zero $\frac{\bar{x}+\underline{x}}{2}=0$. We have

$$v_1^h(x)=\begin{cases}\sqrt{\frac{1}{2(\bar{x}-x^*)}}\ \sin\left(\left[\frac{|x|-x^*}{\bar{x}-x^*}\right]2\pi\right) & \text{if } x\in[\underline{x},x^*) \text{ or } x\in(-x^*,\bar{x}] \\ \sqrt{\frac{1}{2(\bar{x}-x^*)}}\left(\sin\left(\left[\frac{x-x^*}{\bar{x}-x^*}\right]2\pi\right)+\sin\left(\left[\frac{-x-x^*}{\bar{x}-x^*}\right]2\pi\right)\right) & \text{if } x\in[x^*,-x^*]\,.\end{cases}$$

Since the eigenfunction v_1^h is even we use it to construct a projection for the even component of the initial condition $\hat{p}(x)$. For the initial condition in equation (**??**) this gives (for $\alpha<1/2$)

$$\hat{p}^s(x)=\begin{cases}\frac{1-2\alpha}{(\bar{x}-\underline{x})^2\alpha(1-\alpha)} & \text{for} \quad x\in(\underline{x},x^*)\cup(-x^*,\bar{x}) \\ \frac{-1}{(\bar{x}-\underline{x})^2(1-\alpha)} & \text{for } x\in(x^*,-x^*)\end{cases}$$

Therefore, the projection of $\hat{p}^s$ on v_1^h is:

$$b_1^v[\hat{p}^s]=\frac{\int_{\underline{x}}^{\bar{x}}v_1^h(x)\hat{p}^s(x)dx}{\int_{\underline{x}}^{\bar{x}}v_1^h(x)v_1^h(x)dx}=\frac{\int_{\underline{x}}^{0}v_1^h(x)\hat{p}^s(x)dx}{\int_{\underline{x}}^{0}v_1^h(x)v_1^h(x)dx}$$

where the last equality uses that the function is symmetric. Some calculus gives

$$b_1^v[\hat{p}^s]=\frac{2\sqrt{\frac{(1-\alpha)^3}{\bar{x}^3\alpha^2}}\left(\cos\left(\frac{2\pi\alpha}{1-\alpha}\right)-1\right)}{4\pi\alpha-2(1-\alpha)\sin\left(\frac{2\pi\alpha}{1-\alpha}\right)+4\pi(2\alpha-1)\left(\cos\left(\frac{2\pi\alpha}{1-\alpha}\right)-1\right)}\ .$$

Notice therefore that the impulse response in equation (**??**) can be approximated as follows. Rewrite the initial condition as $\hat{p}=\hat{p}^s+\hat{p}^a$ (normalize $\delta=1$), respectively the even and odd component. The odd projection only uses the odd eigenfunctions, so it is

$$\hat{p}^a(x)=\sum_{i=1}^{\infty}b_j^m[\hat{p}^a]\varphi_j^m(x) \quad , \quad \text{where} \quad b_j^m[\hat{p}^a]=\sqrt{\frac{8}{(\bar{x}-\underline{x})^3}}\frac{[\sin(\alpha\pi j)]^2}{\pi j\alpha(1-\alpha)}$$

Using the above result, we approximate the projection for the even component $\hat{p}^s$ as

$$\hat{p}^s(x)\approx b_1^v[\hat{p}^s]v_1^h(x)=b_1^v[\hat{p}^s]\left(\varphi_1^h(x)-\sum_{j=1}^{\infty}\gamma_{1,j}^m\varphi_j^m(x)\right)$$

so the approximate impulse response is

$$H(t)\approx\int_{\underline{x}}^{\bar{x}}\left(e^{\lambda_1^h t}b_1^v[\hat{p}^s]\varphi_1^h(x)+\sum_{j=1}^{\infty}e^{\lambda_j^m t}\left(b_j^m[\hat{p}^a]-b_1^v[\hat{p}^s]\gamma_{1,j}^m\right)\varphi_j^m(x)\right)f(x)dx$$

and o), constructed as $f^s(x)=(f(x)+f(-x))/2$ and $f^o(x)=(f(x)-f(-x))/2$.

or, developing the inner products

$$H(t) \approx e^{\lambda_1^h t} b_1^v[\hat{p}^s] b_1^h[f] + \sum_{j=1}^{\infty} e^{\lambda_j^m t} \left(b_j^m[\hat{p}^a] - b_1^v[\hat{p}^s] \gamma_{1,j}^m \right) b_j^m[f] \tag{24}$$

K Proofs for the Multiproduct model of Appendix ??

Proof. (of Lemma **??**) Using Ito's lemma we have: $dx = (1/2)y^{-1/2}dy - (1/2)(1/4)y^{-3/2}dy^2$ which gives

$$dx = \frac{n-1}{2\,x}dt + dW^a$$

and $w = f(y,z) = z/\sqrt{ny}$. We have:

$$dw = f_y dy + f_z dz + \frac{1}{2} f_{yy}(dy)^2 + \frac{1}{2} f_{zz}(dz)^2 + f_{yz} dy dz$$

where $f = (z/\sqrt{n})\, y^{-1/2}$, and thus

$$\begin{aligned} f_y &= -\frac{z}{2\sqrt{n}} y^{-3/2} \\ f_z &= \frac{1}{\sqrt{n}} y^{-1/2} \\ f_{yy} &= \frac{3z}{4\sqrt{n}} y^{-5/2} \\ f_{zz} &= 0 \\ f_{yz} &= -\frac{1}{2\sqrt{n}} y^{-3/2} \end{aligned}$$

We thus have:

$$\begin{aligned} dw = &-\frac{z}{2\sqrt{n}} y^{-3/2} \left(n dt + 2\sqrt{y}\, dW^a \right) \\ &+ \frac{1}{\sqrt{n}} y^{-1/2} \sqrt{n} \left(\frac{z}{\sqrt{ny}}\, dW^a + \sqrt{1 - \left(\frac{z}{\sqrt{ny}} \right)^2}\, dW^b \right) \\ &+ \frac{1}{2} \frac{3z}{4\sqrt{n}} y^{-5/2} 4y dt - \frac{1}{2\sqrt{n}} y^{-3/2} 2z dt \end{aligned}$$

which we can rearrange as:

$$
\begin{aligned}
dw = & \frac{z}{\sqrt{n}} y^{-3/2} \left(\frac{1-n}{2} \right) dt \\
& + \left(\frac{1}{\sqrt{n}} y^{-1/2} \sqrt{n} \frac{z}{\sqrt{n}\sqrt{y}} - \frac{z}{2\sqrt{n}} y^{-3/2} 2\sqrt{y} \right) dW^a \\
& + \frac{1}{\sqrt{n}} y^{-1/2} \sqrt{n} \sqrt{1 - \left(\frac{z}{\sqrt{n}\sqrt{y}} \right)^2} \, dW^b
\end{aligned}
$$

or

$$
\begin{aligned}
dw &= \frac{w}{x^2} \left(\frac{1-n}{2} \right) dt + \left(\frac{z}{\sqrt{ny}} - \frac{z}{\sqrt{ny}} \right) dW^a + \frac{1}{x} \sqrt{1 - (w)^2} \, dW^b \\
&= \frac{w}{x^2} \left(\frac{1-n}{2} \right) dt + \frac{1}{x} \sqrt{1 - w^2} \, dW^b
\end{aligned}
$$

□

Proof. (of Proposition **??**) We try a multiplicative solution of the form:

$$
\varphi(w, x) = h(w)\, g(x)
$$

To simplify the proof we set $\sigma^2 = 1$. Thus

$$
\begin{aligned}
\lambda h(w) g(x) = & h(w) g'(x) \left(\frac{n-1}{2x} \right) + h'(w) g(x) \frac{w}{x^2} \left(\frac{1-n}{2} \right) \\
& + \frac{1}{2} h''(w) g(x) \frac{(1-w^2)}{x^2} + \frac{1}{2} h(w) g''(x)
\end{aligned}
$$

Dividing by $h(w)$ in both sides we have:

$$
\begin{aligned}
\lambda g(x) = & g'(x) \left(\frac{n-1}{2x} \right) + \frac{h'(w) w}{h(w)} \frac{g(x)}{x^2} \left(\frac{1-n}{2} \right) \\
& + \frac{1}{2} \frac{h''(w)\,(1-w^2)}{h(w)} \frac{g(x)}{x^2} + \frac{1}{2} g''(x)
\end{aligned}
$$

Collecting terms:

$$
\begin{aligned}
\lambda g(x) = & \frac{g(x)}{x^2} \left[\frac{h'(w) w}{h(w)} \left(\frac{1-n}{2} \right) + \frac{1}{2} \frac{h''(w)\,(1-w^2)}{h(w)} \right] \\
& + g'(x) \left[\frac{n-1}{2x} \right] + \frac{1}{2} g''(x)
\end{aligned}
$$

Which suggests to try the following separating variable:

$$
\mu = \frac{h'(w) w}{h(w)} \left(\frac{1-n}{2} \right) + \frac{1}{2} \frac{h''(w)\,(1-w^2)}{h(w)}
$$

or

$$0 = -2\mu h(w) + h'(w)w\,(1-n) + h''(w)\left(1-w^2\right)$$

The solution of this equation is given by the Gegenbauer polynomials $C_m^{\alpha}(w)$. The Gegenbauer polynomials are the solution to the following o.d.e.:

$$(1-w^2)h(w)'' - (2\alpha+1)wh'(w) + m(m+2\alpha)h(w) = 0 \text{ for } w \in [-1,1]$$

for integer $m \geq 0$. Matching coefficients we have:[11]

$$-2\mu = m(m+2\alpha) \text{ and } -(2\alpha+1) = (1-n)$$

which gives

$$\alpha = \frac{n}{2} - 1 \text{ and } \mu = -\frac{m}{2}(m+n-2)$$

Then given $\mu = -(m/2)(m+n-2)$ the o.d.e. for g is:

$$\lambda g(x) = \frac{g(x)}{x^2}\mu + g'(x)\left[\frac{n-1}{2x}\right] + \frac{1}{2}g''(x)$$

or

$$0 = g(x)\left(\mu - x^2\lambda\right) + g'(x)\left[\frac{n-1}{2}\right]x + \frac{1}{2}g''(x)x^2$$

or

$$0 = g(x)\left(2\mu - x^2 2\lambda\right) + g'(x)x\,(n-1) + g''(x)x^2$$

with boundary condition $g(\bar{x}) = 0$. The solution of this o.d.e., which does not explode at $x = 0$ is given by a Bessel function of the first kind. This is because the following o.d.e.:

$$g(x)(c+bx^2) + g'(x)x\,a + g''(x)x^2 = 0$$

has solution:[12]

$$g(x) = x^{(1-a)/2} J_{\nu}\left(\sqrt{b}\,x\right) \text{ where } \nu = \frac{1}{2}\sqrt{(1-a)^2 - 4c}$$

where $J_{\nu}(\cdot)$ is the Bessel function of the first kind. Matching coefficients we have:

$$a = n-1,\ b = -2\lambda,\ c = 2\mu \text{ and}$$
$$\nu = \frac{1}{2}\sqrt{(n-2)^2 - 8\mu} = \frac{1}{2}\sqrt{(n-2)^2 + 8(m/2)(m+n-2)} = \frac{n}{2} - 1 + m$$

[11]See https://en.wikipedia.org/wiki/Gegenbauer_polynomials, which is bases on Abramowitz, Milton; Stegun, Irene Ann, eds. (1983) [June 1964], Chapter 22, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables. Applied Mathematics Series. 55, Dover Publications.

[12]See http://eqworld.ipmnet.ru/en/solutions/ode/ode0215.pdf which uses Polyanin, A. D. and Zaitsev, V. F., Handbook of Exact Solutions for Ordinary Differential Equations, 2nd Edition, Chapman & Hall/CRC, Boca Raton, 2003.

We argue that $\nu = n/2 - 1 + m$ to see that note we have

$$4\nu^2 = (n-2)^2 + 4m(m+n-2) \text{ and}$$
$$4\nu^2 = 4\left(\frac{n-2+2m}{2}\right)^2 = (n-2)^2 + 4m(n-2) + 4m^2$$

which verifies the equality. So we have:

$$g(x) = x^{1-n/2} J_{\frac{n}{2}-1+m}\left(\sqrt{-2\lambda}\, x\right)$$

We still have to determine the eigenvalue λ. For this we use the boundary condition $g(\bar{x}) = 0$ and that $J_\nu(\cdot)$ has infinitely strictly orderer positive zeros, denoted by $j_{\nu,k}$ for $k = 1, 2, \ldots$ so that $J_\nu(j_{\nu,k}) = 0$. Thus fixing μ, i.e. $m \geq 0$, we have:

$$0 = g(\bar{x}) = (\bar{x})^{1-n/2} J_{\frac{n}{2}-1+m}\left(\sqrt{-2\lambda}\, \bar{x}\right)$$

so that:

$$0 = (\bar{x})^{1-n/2} J_{\frac{n}{2}-1+m}\left(\sqrt{-2\,\lambda_{m,k}}\, \bar{x}\right)$$

Hence

$$j_{\frac{n}{2}-1+m,k} = \sqrt{-2\,\lambda_{m,k}}\, \bar{x} \quad \text{or} \quad \lambda_{m,k} = -\frac{\left(j_{\frac{n}{2}-1+m,k}\right)^2}{2\,\bar{x}^2}$$

Collecting the terms for h, g and λ we obtain the desired result.

Since $\sigma^2 \neq 1$ changes the units of time, we need only to adjust the eigenvalue by its value, so that

$$\lambda_{m,k} = -\sigma^2 \frac{\left(j_{\frac{n}{2}-1+m,k}\right)^2}{2\,\bar{x}^2}$$

Using that $N = n\sigma^2/\bar{x}^2$ we get

$$\lambda_{m,k} = -\frac{n\sigma^2}{\bar{x}^2} \frac{\left(j_{\frac{n}{2}-1+m,k}\right)^2}{2n} = N \frac{\left(j_{\frac{n}{2}-1+m,k}\right)^2}{2n}$$

□

Proof. (of Proposition **??**) First take $f(w,x) = w\,x/\sqrt{n} = \frac{1}{n}\sum_{i=1}^n p_i$. But note that the Gegenbauer polynomial of degree 1 is

$$C_1^{\frac{n}{2}-1}(w) = \sum_{k=0}^{\lfloor 1/2 \rfloor} (-1)^k \frac{\Gamma(1-k+\frac{n}{2}-1)}{\Gamma(\frac{n}{2}-1)k!(1-2k)!}(2w)^{1-2k} = \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2}-1)}(2w) = (n-2)w$$

Thus for $f(w,x) = wx/\sqrt{n}$ we can simply write:

$$f(x,w) = \sum_{k=1}^{\infty} b_{1,k}[f]\varphi_{1,k}(x,w)$$

since

$$b_{m,k}[f] = \frac{1}{\sqrt{n}(n-2)} \frac{\left[\int_{-1}^{1} C_1^{\frac{n}{2}-1}(w) C_m^{\frac{n}{2}-1}(w)(1-w^2)^{\frac{n-3}{2}} dw\right] \left[\int_0^{\bar{x}} x\, J_{m+\frac{n}{2}-1}\left(j_{m+\frac{n}{2}-1,k}\frac{x}{\bar{x}}\right) x^{\frac{n}{2}} dx\right]}{\left[\int_{-1}^{1} \left(C_m^{\frac{n}{2}-1}(w)\right)^2 (1-w^2)^{\frac{n-3}{2}} dw\right] \left[\int_0^{\bar{x}} \left(J_{m+\frac{n}{2}-1}\left(j_{m+\frac{n}{2}-1,k}\frac{x}{\bar{x}}\right)\right)^2 x\, dx\right]}$$

and thus $b_{m,k}[f] = 0$ for all $m \neq 1$, since the polynomials are orthogonal, and

$$b_{1,k}[f] = \frac{1}{\sqrt{n}(n-2)} \frac{\int_0^{\bar{x}} x J_{\frac{n}{2}}\left(j_{\frac{n}{2},k}\frac{x}{\bar{x}}\right) x^{\frac{n}{2}} dx}{\int_0^{\bar{x}} \left(J_{\frac{n}{2}}\left(j_{\frac{n}{2},k}\frac{x}{\bar{x}}\right)\right)^2 x\, dx} \text{ for all } k \geq 1$$

Now we argue that fixing x the function $p(w, x; 0)$ is odd (antisymmetric) viewed as a function of w. This is because $\bar{h}$ is even and $x'(0)$ is odd, so $\bar{h}'(w)x'(0)$ is odd. Also $\bar{h}'$ is odd and $w'(0)$ is even, hence $\bar{h}'(w)w'(0)$ is odd. Hence $p(w, x; 0)$ is not orthogonal to the $C_1^{\frac{n}{2}-1}(\cdot)$. Thus $b_{1,k}[\bar{p}'] \neq 0$.

Finally, to represent the survival function, take $f(w, x) = 1$. Note that this also coincides with a Gegenbauer polynomial for $m = 0$, i.e. $C_0^{\frac{n}{2}-1}(w) = 1$. Thus:

$$f(x, w) = \sum_{k=1}^{\infty} b_{0,k}[f]\varphi_{1,k}(x, w)$$

since

$$b_{m,k}[f] = \frac{\left[\int_{-1}^{1} C_0^{\frac{n}{2}-1}(w) C_m^{\frac{n}{2}-1}(w)(1-w^2)^{\frac{n-3}{2}} dw\right] \left[\int_0^{\bar{x}} J_{m+\frac{n}{2}-1}\left(j_{m+\frac{n}{2}-1,k}\frac{x}{\bar{x}}\right) x^{\frac{n}{2}} dx\right]}{\left[\int_{-1}^{1} \left(C_m^{\frac{n}{2}-1}(w)\right)^2 (1-w^2)^{\frac{n-3}{2}} dw\right] \left[\int_0^{\bar{x}} \left(J_{m+\frac{n}{2}-1}\left(j_{m+\frac{n}{2}-1,k}\frac{x}{\bar{x}}\right)\right)^2 x\, dx\right]}$$

and since the Gegenbauer polynomials are orthogonal, and thus $b_{m,k}[f] = 0$ for all $m > 0$, and

$$b_{0,k}[f] = \frac{\int_0^{\bar{x}} J_{\frac{n}{2}-1}\left(j_{\frac{n}{2}-1,k}\frac{x}{\bar{x}}\right) x^{\frac{n}{2}} dx}{\left[\int_0^{\bar{x}} \left(J_{\frac{n}{2}-1}\left(j_{\frac{n}{2}-1,k}\frac{x}{\bar{x}}\right)\right)^2 x\, dx\right]} \text{ for all } k \geq 1$$

□

Proof. (of Proposition ??) Recall that for each $k \geq 1$:

$$\lambda_{1,k} = -N\frac{\left(j_{\frac{n}{2},k}\right)^2}{2n} \text{ and } \lambda_{0,k} = -N\frac{\left(j_{\frac{n}{2}-1,k}\right)^2}{2n}$$

and use $\nu = n/2$ in the first case and $\nu = n/2 - 1$ in the second. It is well know that $j_{\nu,k}$ is strictly increasing in both variables –see Elbert (2001). From here we know that $|\lambda_{1,k}| - |\lambda_{0,k}| > 0$ for all n and k. Also in Elbert (2001) we see that $\frac{\partial}{\partial v} j_{v,k} < 0$ for $\nu > -k$ and $k \geq 1/2$. Thus, the difference between $|\lambda_{1,k}| - |\lambda_{0,k}|$ is decreasing in n.

From Qu and Wong (1999) we have the lower and upper bound for the zeros of the Bessel function $J_\nu(\cdot)$:

$$\nu + \nu^{1/3} 2^{-1/3} |a_k| \leq j_{\nu,k} \leq \nu + \nu^{1/3} 2^{-1/3} |a_k| + \frac{3}{20} |a_k|^2 2^{1/3} \nu^{-1/3}$$

where a_k is the k^{th} zero of the Airy function. Thus, as $n \to \infty$ then $v \to \infty$ and thus both $\lambda_{1,k}$ and $\lambda_{0,k}$ diverge towards $-\infty$. From the same bounds we see that as $n \to \infty$, the difference $\lambda_{0,k} - \lambda_{1,k} \to 1/2$.

□

Proof. (of Proposition **??**) We start with the projections for $z/n = f(w,x) = wx/\sqrt{n}$. We are looking for:

$$\begin{aligned} f(x,w) = wx/\sqrt{n} \sim \sum_{k=1}^{\infty} b_{1,k}[f] \varphi_{1,k}(x,w) &= \sum_{k=1}^{\infty} b_{1,k}[f] h_1(w) g_{1,k}(x) \\ &= \sum_{k=1}^{\infty} b_{1,k}[f]\, C_1^{\frac{n}{2}-1(w)}\, J_{\frac{n}{2}} \left(j_{\frac{n}{2},k} \frac{x}{\bar{x}} \right) x^{1-\frac{n}{2}} \\ &= w\, x^{1-\frac{n}{2}} \, (n-2) \sum_{k=1}^{\infty} b_{1,k}[f] J_{\frac{n}{2}} \left(j_{\frac{n}{2},k} \frac{x}{\bar{x}} \right) \end{aligned}$$

We can replace the expression we obtain below for $b_{1,k}[f]$ to get:

$$\begin{aligned} \sum_{k=1}^{\infty} b_{1,k}[f] \varphi_{1,k}(x,w) &= w\, x^{1-\frac{n}{2}} \, (n-2) \sum_{k=1}^{\infty} \frac{2\, \bar{x}^{\frac{n}{2}}}{\sqrt{n}(n-2)\, j_{\frac{n}{2},k}\, J_{\frac{n}{2}+1} \left(j_{\frac{n}{2},k} \right)} J_{\frac{n}{2}} \left(j_{\frac{n}{2},k} \frac{x}{\bar{x}} \right) \\ &= \frac{w\, x}{\sqrt{n}} \sum_{k=1}^{\infty} \frac{2\, (x/\bar{x})^{-\frac{n}{2}}\, J_{\frac{n}{2}} \left(j_{\frac{n}{2},k} \frac{x}{\bar{x}} \right)}{j_{\frac{n}{2},k}\, J_{\frac{n}{2}+1} \left(j_{\frac{n}{2},k} \right)} \end{aligned}$$

To get the coefficients we start by computing

$$\begin{aligned} &\int_0^{\bar{x}} J_{\frac{n}{2}} \left(j_{\frac{n}{2},k} \frac{x}{\bar{x}} \right) x^{\frac{n}{2}+1} dx = \left(\frac{\bar{x}}{j_{\frac{n}{2},k}} \right)^{\frac{n}{2}+2} \int_0^{\bar{x}} \left(j_{\frac{n}{2},k} \frac{x}{\bar{x}} \right)^{\frac{n}{2}+1} J_{\frac{n}{2}} \left(j_{\frac{n}{2},k} \frac{x}{\bar{x}} \right) j_{\frac{n}{2},k} \frac{dx}{\bar{x}} \\ &= \left(\frac{\bar{x}}{j_{\frac{n}{2},k}} \right)^{\frac{n}{2}+2} \int_0^{j_{\frac{n}{2},k}} (z)^{\frac{n}{2}+1} J_{\frac{n}{2}} (z)\, dz \end{aligned}$$

Using that

$$\int_a^b z^{\nu+1} J_\nu(z) dz = z^{\nu+1} J_{\nu+1}(z) \Big|_a^b$$

then

$$\int_0^{\bar{x}} J_{\frac{n}{2}}\left(j_{\frac{n}{2},k}\,\frac{x}{\bar{x}}\right) x^{\frac{n}{2}+1}dx = \left(\frac{\bar{x}}{j_{\frac{n}{2},k}}\right)^{\frac{n}{2}+2} \int_0^{j_{\frac{n}{2},k}} (z)^{\frac{n}{2}+1}\, J_{\frac{n}{2}}\,(z)\, dz$$
$$= \left(\frac{\bar{x}}{j_{\frac{n}{2},k}}\right)^{\frac{n}{2}+2} \left(j_{\frac{n}{2},k}\right)^{\frac{n}{2}+1} J_{\frac{n}{2}+1}\left(j_{\frac{n}{2},k}\right)$$

Using that

$$\bar{x}^2 \int_0^{\bar{x}} \frac{x}{\bar{x}} \left[J_\nu\left(j_{\nu,k}\frac{x}{\bar{x}}\right)\right]^2 \frac{dx}{\bar{x}} = \frac{1}{2}\left(\,\bar{x}\, J_{\nu+1}\left(j_{\nu,k}\right)\right)^2 \text{ for all } k \in \{1,2,3\ldots\}$$

we have

$$\int_0^{\bar{x}} \left(J_{\frac{n}{2}}\left(j_{\frac{n}{2},k}\,\frac{x}{\bar{x}}\right)\right)^2 x\, dx = \frac{1}{2}\left(\,\bar{x}\, J_{\frac{n}{2}+1}\left(j_{\frac{n}{2},k}\right)\right)^2$$

Thus:

$$b_{1,k}[f] = \frac{2}{\sqrt{n}(n-2)} \frac{\left(\frac{\bar{x}}{j_{\frac{n}{2},k}}\right)^{\frac{n}{2}+2} \left(j_{\frac{n}{2},k}\right)^{\frac{n}{2}+1} J_{\frac{n}{2}+1}\left(j_{\frac{n}{2},k}\right)}{\left(\,\bar{x}\, J_{\frac{n}{2}+1}\left(j_{\frac{n}{2},k}\right)\right)^2}$$
$$= \frac{2\,\bar{x}^{\frac{n}{2}}}{\sqrt{n}(n-2)\, j_{\frac{n}{2},k}\, J_{\frac{n}{2}+1}\left(j_{\frac{n}{2},k}\right)} \text{ for all } k \geq 1$$

Now we turn to compute: $b_{1,k}[\bar{p}'(\cdot,0)]\langle\varphi 1,k,\varphi 1,k\rangle$. We start deriving an explicit expression for $\bar{p}'(\cdot,0)$. We have

$$\bar{h}(w) = \frac{1}{Beta\left(\frac{n-1}{2},\frac{1}{2}\right)}\;\left(1-w^2\right)^{(n-3)/2} \quad \text{for} \quad w \in (-1\,,\,1)$$
$$\bar{g}(x) = x\;(\bar{x})^{-n}\left(\frac{2n}{n-2}\right)\left[\,\bar{x}^{n-2} - x^{n-2}\right] \qquad \text{for } x \in [0,\bar{x}]$$

$$p(w,x;0) = \bar{h}(w(\delta))\bar{g}(x(\delta)) = \bar{h}(w)\bar{g}(x) + \bar{p}'(w,x;0)\delta + o(\delta) \text{ with}$$
$$\bar{p}'(w,x;0) = \bar{g}(x)\bar{h}'(w)w'(0) + \bar{h}(w)\bar{g}'(x)x'(0)$$

where:

$$\frac{\partial}{\partial\delta}x(\delta)|_{\delta=0} = x'(0) = \sqrt{n}\,w \quad \text{and} \quad \frac{\partial}{\partial\delta}\bar{h}(w(\delta))|_{\delta=0} = \bar{h}'(w)w'(0)$$
$$\frac{\partial}{\partial\delta}w(\delta)|_{\delta=0} = w'(0) = \frac{\sqrt{n}\,(1-w^2)}{x} \quad \text{and} \quad \frac{\partial}{\partial\delta}\bar{g}(x(\delta))|_{\delta=0} = \bar{g}'(x)x'(0)$$

so:

$$
\begin{aligned}
\bar{p}'(w,z;0) &= \bar{g}(x)\bar{h}'(w)w'(0) + \bar{h}(w)\bar{g}'(x)x'(0) \\
&= -\left(\bar{x}\right)^{-n}\left(\frac{2n}{n-2}\right)\left[\bar{x}^{n-2} - x^{n-2}\right]\frac{(n-3)w\left(1-w^2\right)^{(n-3)/2}}{Beta\left(\frac{n-1}{2},\frac{1}{2}\right)}\sqrt{n} \\
&+ \frac{w\left(1-w^2\right)^{(n-3)/2}}{Beta\left(\frac{n-1}{2},\frac{1}{2}\right)}\ \bar{x}^{-n}\left[\left(\frac{2n}{n-2}\right)\left[\bar{x}^{n-2} - x^{n-2}\right] - 2nx^{n-2}\right]\sqrt{n} \\
&= \frac{w\left(1-w^2\right)^{(n-3)/2}}{Beta\left(\frac{n-1}{2},\frac{1}{2}\right)}\frac{\sqrt{n}}{\bar{x}^n}\left(\frac{2n}{n-2}\right)\left[(4-n)\left(\bar{x}^{n-2} - x^{n-2}\right) - 2nx^{n-2}\right] \\
&= \frac{w\left(1-w^2\right)^{(n-3)/2}}{Beta\left(\frac{n-1}{2},\frac{1}{2}\right)}\sqrt{n}\left(\frac{2n}{n-2}\right)\frac{\left[(4-n)\bar{x}^{n-2} - (4+n)x^{n-2}\right]}{\bar{x}^n}
\end{aligned}
$$

We want to compute:

$$
b_{1,k}[\bar{p}'(\cdot,0)/\omega]\langle\varphi_{1,k},\varphi_{1,k}\rangle = \int_0^{\bar{x}}\int_{-1}^{1}\bar{p}'(x,w;0)h_{1,k}(x)g_{m,k}(w)dwdx
$$

So we split the integral in the product of two terms. The first term involves the integral over w given by:

$$
\begin{aligned}
&\int_{-1}^{1}(n-2)w\,\frac{w\left(1-w^2\right)^{(n-3)/2}}{Beta\left(\frac{n-1}{2},\frac{1}{2}\right)}\sqrt{n}\left(\frac{2n}{n-2}\right)\,dw \\
&= \frac{2n\sqrt{n}}{Beta\left(\frac{n-1}{2},\frac{1}{2}\right)}\int_{-1}^{1}w^2\left(1-w^2\right)^{(n-3)/2}\,dw = \frac{2n\sqrt{n}}{Beta\left(\frac{n-1}{2},\frac{1}{2}\right)}\frac{\sqrt{\pi}\,\Gamma(\frac{n-1}{2})}{2\,\Gamma\left(\frac{n}{2}+1\right)} \\
&= \frac{n\sqrt{n}\Gamma\left(\frac{n-1}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\frac{\sqrt{\pi}\,\Gamma(\frac{n-1}{2})}{\Gamma\left(\frac{n}{2}+1\right)} = n\sqrt{n}\,\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)} = n\sqrt{n}\,\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}
\end{aligned}
$$

where we use that $C_1^{\frac{n}{2}-1}(w) = (n-2)w$, and properties of the $Beta$ and Γ functions.

The second term involves the integral over x and is given by:

$$\begin{aligned}
&\frac{1}{\bar{x}^n}\int_0^{\bar{x}} \left[(4-n)\bar{x}^{n-2}-(4+n)x^{n-2}\right] J_{n/2}\left(j_{\frac{n}{2},k}\frac{x}{\bar{x}}\right) x^{1-\frac{n}{2}}\,dx \\
&= \frac{\bar{x}^{1-\frac{n}{2}}\bar{x}^{n-2}}{\bar{x}^n}\int_0^{\bar{x}} \left[(4-n)-(4+n)\left(\frac{x}{\bar{x}}\right)^{n-2}\right] J_{n/2}\left(j_{\frac{n}{2},k}\frac{x}{\bar{x}}\right)\left(\frac{x}{\bar{x}}\right)^{1-\frac{n}{2}}\,dx \\
&= \bar{x}^{-\frac{n}{2}}\int_0^{\bar{x}} \left[(4-n)-(4+n)\left(\frac{x}{\bar{x}}\right)^{n-2}\right] J_{n/2}\left(j_{\frac{n}{2},k}\frac{x}{\bar{x}}\right)\left(\frac{x}{\bar{x}}\right)^{1-\frac{n}{2}}\,\frac{dx}{\bar{x}} \\
&= \frac{\bar{x}^{-\frac{n}{2}}}{\left(j_{\frac{n}{2},k}\right)^{2-\frac{n}{2}}}(4-n)\int_0^{\bar{x}} J_{n/2}\left(j_{\frac{n}{2},k}\frac{x}{\bar{x}}\right)\left(\frac{j_{\frac{n}{2},k}x}{\bar{x}}\right)^{1-\frac{n}{2}}\,\frac{j_{\frac{n}{2},k}dx}{\bar{x}} \\
&- \frac{\bar{x}^{-\frac{n}{2}}}{\left(j_{\frac{n}{2},k}\right)^{\frac{n}{2}}}(4+n)\int_0^{\bar{x}} \left(\frac{j_{\frac{n}{2},k}x}{\bar{x}}\right)^{n-2} J_{n/2}\left(j_{\frac{n}{2},k}\frac{x}{\bar{x}}\right)\left(\frac{j_{\frac{n}{2},k}x}{\bar{x}}\right)^{1-\frac{n}{2}}\,\frac{j_{\frac{n}{2},k}dx}{\bar{x}} \\
&= \frac{\bar{x}^{-\frac{n}{2}}}{\left(j_{\frac{n}{2},k}\right)^{2-\frac{n}{2}}}(4-n)\int_0^{j_{\frac{n}{2},k}} z^{1-\frac{n}{2}}J_{\frac{n}{2}}(z)\,dz \\
&- \frac{\bar{x}^{-\frac{n}{2}}}{\left(j_{\frac{n}{2},k}\right)^{\frac{n}{2}}}(4+n)\int_0^{j_{\frac{n}{2},k}} z^{\frac{n}{2}-1}J_{\frac{n}{2}}(z)\,dz
\end{aligned}$$

To find an expression for this integrals note that:

$$\int_0^a z^{1-\frac{n}{2}}J_{\frac{n}{2}}(z)\,dz = -\frac{2^{1-n/2}\left(-1+{}_0F_1\left(n/2,-a^2/4\right)\right)}{\Gamma(n/2)} = \frac{2^{1-\frac{n}{2}}}{\Gamma(\frac{n}{2})} - a^{1-\frac{n}{2}}J_{\frac{n}{2}-1}(a)$$

and

$$\int_0^a z^{\frac{n}{2}-1}J_{\frac{n}{2}}(z)\,dz = 2^{-1-\frac{n}{2}}a^n\Gamma\left(\frac{n}{2}\right){}_1\tilde{F}_2\left(\frac{n}{2};1+\frac{n}{2},1+\frac{n}{2};-\frac{a^2}{4}\right)$$

where ${}_1\tilde{F}_2(a_1;b_1,b_2;z)$ is the regularized generalized hypergeometric function, i.e. it is defined as ${}_1\tilde{F}_2(a_1;b_1,b_2;z) = {}_1F_2(a_1;b_1,b_2;z)/\left(\Gamma(b_1)\Gamma(b_2)\right)$ where ${}_1F_2$ is the generalized hypergeometric function. Thus

$$\begin{aligned}
&\frac{1}{\bar{x}^n}\int_0^{\bar{x}} \left[(4-n)\bar{x}^{n-2}-(4+n)x^{n-2}\right] J_{n/2}\left(j_{\frac{n}{2},k}\frac{x}{\bar{x}}\right) x^{1-\frac{n}{2}}\,dx \\
&= \bar{x}^{-\frac{n}{2}}\left[\frac{(4-n)}{\left(j_{\frac{n}{2},k}\right)^{2-\frac{n}{2}}}\left(\frac{2^{1-\frac{n}{2}}}{\Gamma(\frac{n}{2})} - (j_{\frac{n}{2},k})^{1-\frac{n}{2}}J_{\frac{n}{2}-1}\left(j_{\frac{n}{2},k}\right)\right)\right. \\
&\left. -\frac{(4+n)}{\left(j_{\frac{n}{2},k}\right)^{\frac{n}{2}}}2^{-1-\frac{n}{2}}(j_{\frac{n}{2},k})^n\Gamma\left(\frac{n}{2}\right){}_1\tilde{F}_2\left(\frac{n}{2};1+\frac{n}{2},1+\frac{n}{2};-\frac{(j_{\frac{n}{2},k})^2}{4}\right)\right]
\end{aligned}$$

Thus we have:

$$
\begin{aligned}
&b_{1,k}[f]b_{1,k}[\bar{p}'(\cdot,0)]\langle\varphi_{1,k},\varphi_{1,k}\rangle \\
&= n\sqrt{n}\,\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}\frac{2\,\bar{x}^{\frac{n}{2}}}{\sqrt{n}(n-2)\,j_{\frac{n}{2},k}\,J_{\frac{n}{2}+1}\left(j_{\frac{n}{2},k}\right)} \\
&\bar{x}^{-\frac{n}{2}}\left[\frac{(4-n)}{\left(j_{\frac{n}{2},k}\right)^{2-\frac{n}{2}}}\left(\frac{2^{1-\frac{n}{2}}}{\Gamma(\frac{n}{2})}-(j_{\frac{n}{2},k})^{1-\frac{n}{2}}J_{\frac{n}{2}-1}\left(j_{\frac{n}{2},k}\right)\right)\right. \\
&\left.-\frac{(4+n)}{\left(j_{\frac{n}{2},k}\right)^{\frac{n}{2}}}2^{-1-\frac{n}{2}}(j_{\frac{n}{2},k})^{n}\Gamma\left(\frac{n}{2}\right){}_1\tilde{F}_2\left(\frac{n}{2};1+\frac{n}{2},1+\frac{n}{2};-\frac{(j_{\frac{n}{2},k})^2}{4}\right)\right] \\
&= \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}\frac{2n}{(n-2)\,j_{\frac{n}{2},k}\,J_{\frac{n}{2}+1}\left(j_{\frac{n}{2},k}\right)} \\
&\left[\frac{(4-n)}{\left(j_{\frac{n}{2},k}\right)^{2-\frac{n}{2}}}\left(\frac{2^{1-\frac{n}{2}}}{\Gamma(\frac{n}{2})}-(j_{\frac{n}{2},k})^{1-\frac{n}{2}}J_{\frac{n}{2}-1}\left(j_{\frac{n}{2},k}\right)\right)\right. \\
&\left.-\frac{(4+n)}{\left(j_{\frac{n}{2},k}\right)^{\frac{n}{2}}}2^{-1-\frac{n}{2}}(j_{\frac{n}{2},k})^{n}\Gamma\left(\frac{n}{2}\right){}_1\tilde{F}_2\left(\frac{n}{2};1+\frac{n}{2},1+\frac{n}{2};-\frac{(j_{\frac{n}{2},k})^2}{4}\right)\right] \\
&= \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}\frac{2n}{(n-2)\,j_{\frac{n}{2},k}\,J_{\frac{n}{2}+1}\left(j_{\frac{n}{2},k}\right)} \\
&\left[(4-n)\left(\frac{2^{1-\frac{n}{2}}}{\Gamma(\frac{n}{2})\left(j_{\frac{n}{2},k}\right)^{2-\frac{n}{2}}}-\frac{J_{\frac{n}{2}-1}\left(j_{\frac{n}{2},k}\right)}{j_{\frac{n}{2},k}}\right)\right. \\
&\left.-(4+n)2^{-1-\frac{n}{2}}(j_{\frac{n}{2},k})^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right){}_1\tilde{F}_2\left(\frac{n}{2};1+\frac{n}{2},1+\frac{n}{2};-\frac{(j_{\frac{n}{2},k})^2}{4}\right)\right]
\end{aligned}
$$

□

L Equivalence with discrete-time finite-state models

In this section we collect result or comments that may help understand the result better, but that are not central to them.

L.1 Discrete time, finite state case

This appendix develops the discrete-time, finite-state analogue case of the continuous time problem discussed in the paper.

Suppose we have a state x_t at time t that takes n values $x_t \in \{\xi_1, \xi_2, .., \xi_m\}$. Instead we refer to the realizations at time t as $x_t = \xi_i$. The transition for the measure of x_t is given by an $m \times m$ matrix A with positive elements a_{ij}. The element a_{ij} has the interpretation of the measure that $x_{t+1} = \xi_j$ will have if todays value is $x_t = \xi_i$. Note that we are interested in the case where $\sum_j^n \mathtt{a}_{ij} \leq 1$, so that

$$q_i \equiv \sum_{j=1}^{m} \mathtt{a}_{ij} \leq 1$$

has the interpretation of the surviving probability of x_{t+1} conditional to $x_t = \xi_i$. We are interested in computing the expected values of a function f of the state using this measure, which we refer as the hat-expectations. Such function can be represented by a vector $f \in \mathbb{R}^m$. We define the expected value conditional on surviving as:

$$\mathcal{G}(f)(\xi_i, 1) \equiv \sum_{j=1}^{m} f_j \, \mathtt{a}_{ij}$$

Given the matrix A we can define $\mathcal{G}(f)$ as:

$$\mathcal{G}(f) \equiv Af \quad \text{or element by element by} \quad \mathcal{G}(f)(\xi_i, 1) \equiv (Af)_i \text{ for all } i = 1, \ldots, m$$

where the Af is an m dimensional vector, and $(Af)_i$ is its ith element. Note that we can iterate in this operator and define the conditional expectations $s \geq 1$ period ahead as:

$$\mathcal{G}(f) \equiv A^s f \quad \text{or element by element by} \quad \mathcal{G}(f)(\xi_i, s) \equiv (A^s f)_i \text{ for all } i = 1, \ldots, m$$

where A^s is the s power of the matrix A.

Not surprisingly, since transitions are given by powers of the function A, it It turns out that the eigenvalues and eigenvectors of the matrix A contain very important information. Each eigenvalue-eigenvector (λ_j, φ_j) satisfies: $\lambda_j \varphi_j = A\varphi_j$. Then, pre-multiplying both sides A we get: $\lambda_j A\varphi_j = A^2\varphi_j$ and using that (λ_j, φ_j) is an eigenvalue-eigenvector pair: $\lambda_j^2 \varphi_j = A^2\varphi_j$. Continue this way we obtain:

$$\lambda_j^t \, \varphi_j = A^t \, \varphi_j \text{ or } = \mathcal{G}(f)(\xi_i, t) = \lambda_j^t \, \varphi_j(\xi_i) \quad \text{for } i = 1, \ldots, m$$

Thus, the eigenvectors are special functions whose conditional hat-expectations are just its current value times its corresponding eigenvalue raised to the time spanned.

We assume that the initial state x_0 is drawn from a distribution p_0, so that $\sum_{i=1}^m p_0(\xi_i) = 1$. We are interested in computing the expected value of the function f at time t, assuming that x_0 has been drawn from p_0. In particular, we want to compute:

$$G(t) = \sum_{i=1}^{m} \mathcal{G}(f)(\xi_i, t) \, p_0(\xi_i) \text{ for all } t \geq 0$$

Besides the entire function F, there are other summary measures of interest. One is cumulative sum, i.e. $\mathbf{G} \equiv \sum_{t=0}^{\infty} G(t)$, and the other is the discrete slope at zero, i.e. $\Delta G(0) \equiv G(1) - G(0)$.

Assume that the vector f that can be written as a linear combination of k such eigenvectors, with coefficients $b_j[f]$

$$f = \sum_{j=1}^{k} b_j[f]\,\varphi_j \text{ or } f(\xi_i) = \sum_{j=1}^{k} b_j[f]\,\varphi_j(\xi_i) \text{ for all } i = 1,\ldots,k \text{ and } k \le m$$

Note that the k eigenvectors with non-zero weight may be a subset of the m dimensions.

Then we can compute the hat-expected value of f at time t when x_0 is drown from p_0 as:

$$\begin{aligned} G(t) &\equiv \sum_{i=1}^{m} \mathcal{G}(f)\,(\xi_i,t)\,p_0(\xi_i) = \sum_{i=1}^{m} \mathcal{G}\left(\sum_{j=1}^{k} b_j[f]\varphi_j\right)(\xi_i,t)\ \ p_0(\xi_i) \\ &= \sum_{i=1}^{m}\sum_{j=1}^{k} b_j[f]\mathcal{G}\left(\varphi_j\right)(\xi_i,t)\,p_0(\xi_i) = \sum_{j=1}^{k} b_j[f]\sum_{i=1}^{m}\mathcal{G}\left(\varphi_j\right)(\xi_i,t)\,p_0(\xi_i) \\ &= \sum_{j=1}^{k} \lambda_j^t\, b_j[f]\,\sum_{i=1}^{m}\varphi_j(\xi_i)\,p_0(\xi_i) = \sum_{j=1}^{k}\lambda_j^t\, b_j[f]\, b_j[p_0] \end{aligned}$$

where we have defined $b_j[p_0]$ as:

$$b_j[p_0] \equiv \sum_{i=1}^{m} \varphi_j(\xi_i)p_0(\xi_i) \quad \text{for all } j = 1,\ldots,k$$

Note also that $\mathbf{G}$ and $F'(0)$ are given by:

$$\begin{aligned} \mathbf{G} &\equiv \sum_{t=0}^{\infty} G(t) = \sum_{j=1}^{k} \frac{b_j[f]\,b_j[p_0]}{1-\lambda_j} \\ \Delta G(0) &\equiv G(1) - G(0) = \sum_{j=1}^{k} b_j[f]\,b_j[p_0]\,[\lambda_j - 1] \end{aligned}$$

If the eigenvectors $\{\varphi_j\}$ that we use are orthogonal, and that they are normalized so they have norm equal to one, i.e.:

$$\begin{aligned} 0 &= \sum_{i=1}^{k} \varphi_j(\xi_i)\varphi_r(\xi_i) \quad \text{for all } j,r = 1,\ldots,k \text{ and} \\ 1 &= \sum_{i=1}^{k} \left[\varphi_j(\xi_i)\right]^2 \quad \text{for all } j = 1,\ldots,k \end{aligned}$$

then the coefficients $b_j[\cdot]$ have the interpretation of a projection. This makes its manipulation and interpretation much easier. Of course if one knows all the eigenvectors, and they form a base, any function f can be written as a linear combination of the eigenvectors. A related issue is whether the eigenvalues and eigenvectors are real or complex.

Finally, since the matrix A is positive, the Perron-Frobenius theorem shows that the largest eigenvalue is smaller or equal to one, and that it is the only whose associated eigenvalue has all positive elements.[13] Thus, without loss of generality, we will order the eigenvalues as $|\lambda_1| > |\lambda_2| \ldots$. The pair (λ_1, φ_1) is called the dominant eigenvalue-eigenvector, since they describe the slowest rate of convergence of the state, i.e. provided that $b_1[f]b_1[p_0] \neq 0$ we have that:

$$\lim_{t\to\infty} \frac{\log |G(t)|}{t} = \log \lambda_1$$

If $b_1[f]b_1[p_0] \neq 0$. Otherwise, the same expression hold for the eigenvalue i^* where $|\lambda_i|$ is closest to one among those where $b_i[f]b_i[p_0] \neq 0$.

To see this, add and subtract $\log\left(|\lambda_1^t b_1[f]\, b_1[p_0]|\right)$ to $\log G(t)$, and divide by t obtaining:

$$\frac{\log |G(t)|}{t} = \log \lambda_1 + \frac{1}{t}\left[\log\left(|b_1[f]\, b_1[p_0]|\right) + \log\left(\frac{\sum_{j=1}^k \lambda_j^t\, b_j[f]\, b_j[p_0]}{\lambda_1^t\, b_1[f]\, b_1[p_0]}\right)\right]$$

Taking the limit we obtain the desired result, provided that $|b_1[f]\, b_1[p_0]| > 0$.

There is also a parallel very closely analysis of the dynamics of the distribution p_t, which we relegate to an appendix.

L.2 Discrete-time, finite state examples

Here we discuss ways to compute the expectation of the function f at time t conditional on the state x being drawn form distribution p_0 at time $t = 0$. This discussion mimics the linear operator notation used in the general case.

We also consider the related problem of the transition for the measure of the states at t, denoted by $p_t \in \mathbb{R}^m_+$, to the measure of the states at $t+1$, denoted by $p_{t+1} \in \mathbb{R}^m_+$. In particular we let $p_t(i)$ denote the measure at t of state $x_t = \xi_i$. This transition for this measure is given by:

$$p_{t+1} = B\, p_t = A'\, p_t \;\text{ or element by element by }\; p_{t+1}(i) = \sum_{j=1}^m \mathtt{b}_{i,j}\, p_t(j) = \sum_{j=1}^m \mathtt{a}_{j,i}\, p_t(j) \text{ for all } i = 1, \ldots, m$$

and thus $B = A'$, is the transpose of the matrix A, so that $\mathtt{b}_{ij} = \mathtt{a}_{ji}$.

Note that we can iterate in this operator and define the values of the measure p_{t+s} after $s \geq 1$ periods as:

$$p_{t+s} = B^s\, p_t \;\text{ or element by element by }\; p_{t+1}(i) = (B^s p_t)_i \text{ for all } i = 1, \ldots, m$$

As an intermediate step, we compute the 'hat'-expectation of the function of the state $f(x_{t+1})$, conditional on x_t being drawn from the measure p_t. We can do this in two different ways. The first way is to compute the conditional expectations, and then average the states x_t using p_t. The second is to compute the hat-expectation with the measure p_{t+1} implied by

[13] The exact nature of the statement depends on whether all the elements are strictly positive or just non-negative.

the law of motion B and the initial measure p_t. The first way is:

$$\sum_{i=1}^{n} \hat{\mathbb{E}}\left[f(x_{t+1})|x_t = \xi_i\right] p_t(i) = \langle Af \, , \, p_t \rangle = (Af)' p_t$$

where the symbol $\langle \cdot, \cdot \rangle$ denotes inner product, so that $\langle g, h \rangle = g'h$ for any two $n-$dimensional vectors g, h.

The second way of computing this expectation is

$$\sum_{i=1}^{n} f_i \, p_{t+1}(i) = \sum_{i=1}^{m} f_i \, (Bp_t)_i = \langle f \, , \, Bp_t \rangle = f'(Bp_t)$$

To see why these two calculations give the same answer note that:

$$\langle f \, , \, Bp_t \rangle = f'(Bp_t) = (Bp_t)'f = p'B'f = p_t'(Af) = (Af)'p_t = \langle Af \, , \, p_t \rangle$$

where the first equality follows from definition of inner product, the second and third by transposing the elements, the fourth by definition of B, the fifth by transposing the elements, and the last one by definition of the inner product. Thus we have:

$$\langle f \, , \, Bp_t \rangle = \langle Af \, , \, p_t \rangle \text{ for any } \; p_t \in \mathbb{R}^m_+ \text{ and } f \in \mathbb{R}^m$$

This is indeed the definition that the operators A and B are self-adjoints.

We can use the same procedure for the hat-expectation t periods ahead starting from p_0 and obtain:

$$\langle f \, , \, B^t p_0 \rangle = \langle A^t f \, , \, p_0 \rangle \text{ for any } \; p_t \in \mathbb{R}^m_+ \text{ and } f \in \mathbb{R}^m$$

Finally, to compute A^t, the t powers of matrix A, it is convenient to diagonalize the matrix A. We proceed under the assumption that this matrix has all distinct real eigenvalues λ_i for $i = 1, \ldots, m$, which we collect in the diagonal matrix Λ. We let Φ be the matrix with the corresponding eigenvectors:

$$A = \Phi^{-1}\Lambda\Phi \implies \Phi A = \Lambda\Phi \implies A'\Phi' = \Phi'\Lambda \implies B\Phi' = \Phi'\Lambda$$

Suppose there is an n dimensional vector $b[f]$ for which

$$f = \Phi \, b[f] \text{ or element by element by } \; f_i = \sum_{j=1}^{m} \varphi_{ij} \, b_j[f] \text{ for all } i = 1, \ldots, m$$

Then we can write:

$$\begin{aligned}
\langle A^t f \, , \, p_0 \rangle &= \langle f \, , \, B^t p_0 \rangle = \langle \Phi b[f] \, , \, B^t p_0 \rangle = \langle \Phi b[f] \, , \, (\Phi')^{-1}\Lambda^t \Phi' p_0 \rangle \\
&= b[f]'\Phi'(\Phi')^{-1}\Lambda^t(\Phi' p_0) = b[f]'\Lambda^t b[p_0] \\
&= \sum_{i=1}^{n} (\lambda_i)^t \; b_i[f] \, b_i[p_0]
\end{aligned}$$

where we denote by $b[p_0]$ the $n-$dimensional vector for which $p_0 = \Phi\ b[p_0]$. For the record, $b[f] = \Phi^{-1} f$ and $b[p_0] = \Phi^{-1} p_0$

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