Working paper

PRECAUTION, INFORMATION AND TIME-INCONSISTENCY: ON THE VALUE OF THE *PRECAUTIONARY PRINCIPLE* [1]

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Abstract. The *Precautionary Principle* is a controversial policy instrument, often criticized for stifling innovation and growth. In this paper, we introduce a model of risky technology reflecting real-life situations where policymakers have called for and sometimes implemented the *Precautionary Principle*. We define this *Principle* as an institutional cap on actions that cannot be adjusted for a fixed period of time and ask whether it is valuable, and under which circumstances, to impose such cap. If he starts using the technology, a decision-maker faces the possibility of an irreversible catastrophe, an event that follows a non-homogeneous Poisson process with a rate that depends on the stock of past actions. Passed a tipping point, the rate increases. We describe optimal trajectories under different degrees of knowledge on the tipping point. When the mere fact of having passed the tipping point is immediately known, the optimal action plan is time-consistent, and the *Precautionary Principle* is irrelevant. When having passed the tipping point remains unknown, a scenario of deep uncertainty, a time-inconsistency problem arises. We characterize both the commitment solution and a *Stock-Markov Equilibrium* such that the decision-maker uses at any point in time a feedback rule that depends only on the existing stock of past actions. Imposing a *Precautionary Principle* at the beginning of the period can improve commitment. We prove that such a restriction is optimal when passing the tipping point is unlikely to happen early on, a scenario that would lead decision-makers to increase action levels too quickly.

keywords. *Precautionary Principle*, Regulation, Environmental Risk, Tipping Point, Uncertainty, Time Inconsistency, Functional and Differential Equations.

JEL codes. D83, Q55.

1. INTRODUCTION

On the *Precautionary Principle*. The major environmental and health issues that pertain to our modern *risk society* are often due to our own production or consumption activities.[1] When dealing with such risks, policy decision-making is complicated by two features that make the standard tools of cost-benefit analysis of little value or even irrelevant. The first specificity is that consumption and production choices might entail a strong irreversibility component. The most salient example is given by global warming. Pollutants have been accumulating in the atmosphere from the beginning of the industrial era, leading to a steady increase in temperature. All current or planned efforts against global warming

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[1]See Beck (1992).

consist in controlling the growth rate of temperature, with little hope of reducing it. Another example is given by *GMO* crops whose production may profoundly modify the surrounding biotope without any possibility of engineering back that biotope because of irreversible mutations.[2]

The second feature of those problems is that the costs and benefits of any decisions have to be assessed in a world of significant uncertainty. Although the consequences of acting might be detrimental to the environment, the extent to which it is so and the probability of harmful events, in other words the physical processes at play, remain to a large extent unknown to decision-makers at the time of acting.

The policy guidelines that have been adopted to structure decision-making and regulation in those contexts widely vary from one country to the other. To illustrate, while *GMOs* are authorized for human consumption in the U.S. without labelling, it is compulsory to label them in sixty four other countries throughout the world and they are actually forbidden in most of the E.U.. Despite such variations in responses, a common concern has been to improve knowledge of the risky phenomenons at stake and thus to let scientific expertise play a significant role throughout the decision-making process.[3]

To further guide decision-making, a concept that has repeatedly been invoked is the so called *Precautionary Principle.* The original idea is due to the philosopher Hans Jonas' *Vorsorgeprinzip*, or *Principle of Foresight* - sometimes translated and referred to as the *Principle of Responsibility.* This *Principle* suggests that we should acknowledge the long-term irreversible consequences of present actions, and refrain from undertaking any such action if there is no proof that it would not negatively affect future generations' well-being. The *Precautionary Principle* was acknowledged by the United Nations in 1992, during the *Earth Summit* held in Rio, and expressed perhaps less restrictively as: *"Where there are threats of serious and irreversible damage, lack of full scientific certainty shall not be used as a reason for postponing cost-effective measures to prevent environmental degradation."* The same idea was then developed and adopted by several other governments. In France, a very similar principle was written in the 2004 *Charter on Environment*,[4] that is now part of the French Constitution. Any risk regulation must comply with the legal framework that the *Principle* contributes to build. In most cases, it takes the form of a law that states a period during which some actions cannot be undertaken, or only at a very limited level. For example, Switzerland voted in 2017 to ban *GMO* cultures for four years[5]. This is the definition of the *Precautionary Principle* we will use throughout this paper.

There has always been a lively debate on whether the *Precautionary Principle* provides a convenient guide for decision-making under deep uncertainty. Doubts exist on the fact that its adoption might actually do more harm, by hindering innovation and wealth creation, than good, by protecting human health or the environment. This debate, mainly

[2]Other examples include hydraulic fracturing to exploit shale gas (which implies irreversible pollution of underground water reserves), authorizing the use of bisphenol A or glyphosate (which are both potential sources of cancers), relying exclusively on nuclear energy (with potential severe environmental destruction and health issues in case of an accident).

[3]We shall leave aside the concerns about the reliability of information and how it can be manipulated or interpreted by groups of different backgrounds and experts. For some related discussion of those considerations, we refer to Hood, Rothstein and Baldwin (2003, Chapter 2).

[4]Loi constitutionnelle n 2005-205 du 1 mars 2005 relative à la Charte de l'environnement.

[5]https://www.letemps.ch/suisse/cultures-ogm-ne-pousseront-sitot-suisse

led by sociologists and political scientists, illustrates the contradictory views that pertains to the *Precautionary Principle*.[6] Giddens (2011) forcefully argues that preventing one risk may sometimes trigger another. A ban on *GMOs* may increase the risk of starvation and malnutrition in an uncertain future. Following the 2011 disaster in Fukushima, powerful interest groups throughout Europe have been advocating for a ban on nuclear energy; but on the other hand, it probably means relying on fossil energies even more at the cost of accelerating global warming. Sunstein (2005) also points out that the *Precautionary Principle* is sometimes understood as meaning *not acting* because more of the act is also associated to a greater harm while *true* precaution might instead require taking serious actions. Fighting global warming is an example in order. Finally, commentators have also wondered about the exact definition of the *Precautionary Principle* which seems to be modified on a case-by-case basis.[7] One example of the fuzziness of the concept is the difficulty to agree on what is meant by *"full scientific certainty"*, or its absence. To illustrate, while the intensity of damages following a nuclear accident is unfortunately perfectly known, it is possible that at some point in the future, advances in science might make the probability of such catastrophe much smaller. But overturning precautionary stances, if written into the Constitution, might be extremely difficult by then.[8]

That ethical considerations have entered the judicial arsenal and how such entry has been perceived by practitioners raises two important comments. The first one is that any ban on acting that the *Principle* recommends is only justified and thus matters if the *laissez-faire* outcome would require, on the contrary, excessive actions. Any *laissez-faire* theory justifying the *Principle* is thus conceptually flawed. In other words, the *Precautionary Principle* can only find some rationale if it solves a conflict of interests between the Constitutional level, which aims to maximize the well-being of present and future generations, and decision-makers who are in charge of implementing actions at every period. Our analysis will unveil such conflict and find its source in the dynamics of actions in a world of deep uncertainty. Time-inconsistency may be a concern in such context. The *Precautionary Principle* may then be viewed as a rough and incomplete social contract[9] that could help to solve a commitment problem.

Model and Results. A risky technology is available. Thanks to this technology, a decision-maker can choose at any point in time an action giving a flow surplus. The past stock of actions affects the arrival rate of an environmental disaster. Following such disaster, viewed as a major disruptive event, opportunities for consumption/production disappear and a flow damage is incurred from that date on. This event follows a non-homogenous Poisson process. To capture the idea that past actions have an irreversible impact, this rate depends on the stock of past actions. More precisely, when the stock reaches a given tipping point, the rate discontinuously jumps upwards.[10]

[6]See Gardiner (2006) and O'Riordan (2013) for informed discussions.

[7]See Immordino (2003) on this.

[8]Austria banned nuclear power in 1978, arguably before greenhouse gases emissions became a strong concern for citizens.

[9]Grossman and Hart (1986).

[10]Tipping points models are frequently used in ecology and in climatology (Lenton et al., 2008). To illustrate, a recent report by the World Bank argues that *"As global warming approaches and exceeds 2-degrees Celsius, there is a risk of triggering nonlinear tipping elements. Examples include the disintegration of the West Antarctic ice sheet leading to more rapid sea-level rise.The melting of the Arctic permafrost ice also induces the release of carbon dioxide, methane and other greenhouse gases which would considerably accelerate global warming."* See http://whrc.org/project/arctic-permafrost.

We consider different scenarios of information learning along the process.

The decision-maker knows where the tipping point lies. This is the simplest scenario. Actions taken early on have now an opportunity cost since they contribute to approaching the tipping point; an *Irreversibility Effect.* Because of discounting and because all actions taken earlier on make the same contribution in coming close to the tipping point, optimal actions are reduced over time during an initial *precautionary phase.* Distortions below the myopic optimum are driven by the sole concern for irreversibility. Once the tipping point has been passed, actions have no longer any impact on the arrival rate. The decision-maker maximizes current benefits by jumping to a higher myopic action. The benefit of low actions early on is to postpone the date at which the tipping point is reached. Yet postponing that date also has a cost since actions can be raised up to the myopic optimum once the tipping point is passed. The *Precautionary Principle* is here irrelevant because, if the project is valuable, a constraint on actions earlier on would only make the decision-maker postpone high actions into the future, resulting in a welfare loss.

The decision-maker only knows when the tipping point has been passed. Suppose now that the tipping point is not *a priori* known. Only the distribution from which it is drawn is known.[11] Yet, the mere fact of having passed the tipping point is immediately learned. This scenario arises when scientific knowledge is sufficiently advanced to figure out such event. A by-product of such information structure is that as long as the tipping point is known not to have been passed, the decision-maker also knows that the arrival rate remains low. From a dynamic programming point of view, the state of the system is thus entirely determined by the stock of past actions. The decision-maker looks for an optimal action plan that prevails as long as ignorance on the value of the tipping point remains. The decision-maker acts accordingly; taking into account the irreversibility of his earlier actions and the uncertainty on where the tipping point lies. Once the tipping point is passed, the decision-maker immediately switches to the myopic action forever just as in the common knowledge scenario.[12]

The dynamic optimization problem has a recursive structure. The *Principle of Dynamic Programming* applies and the solution is time-consistent. We fully characterize the optimal trajectory by means of a Hamilton-Bellman-Jacobi equation for the value function together with a feedback rule that determines how the current action (conditionally on not having yet passed the tipping point) varies with state of the system, i.e., the existing stock of past actions. The *Irreversibility Effect* is still at play. Uncertainty on the location of the tipping point does not change the decision-maker's incentives to reduce actions before it is reached. The trajectory again features some discontinuous jump in actions once it is known that the tipping point has been passed. The *Precautionary Principle* is again irrelevant. Reducing actions earlier on would postpone the choice of higher actions and the crossing of the tipping point into the future, again resulting in a welfare loss.

Deep uncertainty. Suppose now that the decision-maker remains ignorant on whether the

[11] The case of an agnostic Laplace distribution is a particular example of some relevance in practice. Kriegler et al. (2009) offers a view of what experts might think of those distributions of tipping points. Roe and Baker (2007) argues that whether past actions have already triggered a change of regimes might remain unknown for a while.

[12] This scenario bears some resemblance with Loury (1979)'s analysis of how to exploit a resource with unknown reserve. In that model as well, when the decision-maker has reached the limits of the resource stock he immediately knows it and stops consuming from that date on.

tipping point has already been passed. His action can no longer be conditioned on that event. At any point in time t, the decision-maker, ignorant on where the tipping point lies, evaluates his future payoffs with an implicit discount rate that encapsulates the expected probability of survival from that date on. This discount rate evolves along the trajectory as more potential values of the tipping points are crossed and no disaster arises. The marginal rate of substitution between two subsequent dates in the future increases as the decision-maker comes closer to those dates. Ideally, the decision-maker would like to commit to keeping low actions in the future, because doing so would maintain the implicit discount rate at a low level, thus preserving the relative value of future actions. Yet, the decision-maker has incentives to re-optimize his action plan over time and choose higher actions with respect to what was initially planned.

Accordingly, we look for an equilibrium concept that captures this continuous re-optimization. To do so, we characterize a so-called *Stock-Markov Equilibrium* where the decision-maker follows a feedback rule that specifies actions contingent on current stock of past actions. This rule together with an associated *pseudo*-value function[13] allows the decision-maker to commit only for periods of arbitrarily small length. At any point in time, the decision-maker thus chooses an action that is optimal given the current stock of past actions and given that he expects his own selves to stick to the same feedback rule later on, when the stock will have evolved according to his own current choice. The *pseudo*-value function satisfies a functional equation that, although bearing some similarities with a Hamilton-Bellman-Jacobi equation, is now *non-local* in nature. This non-local nature captures the externality that his future selves exert on the decision-maker today.

Characterizing the solution to such a functional equation requires involved techniques. We transform this functional equation into a pair of differential equations respectively for the *pseudo*-value function and the externality component of the payoff. The properties of this system are analyzed by means of the Cauchy-Lipschitz, Wintner and Hartman-Grobman Theorems which provide existence and uniqueness of the time-consistent feedback rule and the *pseudo*-value function under various circumstances. We also analyze the asymptotic behavior of these variables and derive tight bounds. In particular, the equilibrium action (which by definition cannot be conditioned on whether the tipping has been passed or not) and the value function both converge towards their myopic levels. Moreover, the optimal action remains positive. *Not acting* is never optimal.

The Precautionary Principle. This time-inconsistency problem is akin to a conflict of interests between the decision-maker's selves acting at different points in time. It provides a sound foundation for viewing the *Precautionary Principle* as a way of solving this conflict. Committing to a fixed action before more information is learned (which in our context means that it becomes more likely that the tipping point has been passed after this commitment phase) indeed forces future selves to abide to the rule chosen earlier on. Yet, the cost of such commitment is that the action no longer depends on the current stock, i.e., on how much has been learned on the arrival rate of a disaster. The trade-off is of course reminiscent of the *rules versus discretion* debate that arises (under different

[13]The qualifier *pseudo* captures the fact that this value function takes into account that future actions will be taken by the decision-maker with the same equilibrium requirement of time-consistency.

forms) in macroeconomics,[14] political science[15] and mechanism design.[16] In our context, the conflict of interests is between the different selves of a given decision-maker, acting at different points in time in a context of time-inconsistency. We demonstrate that the *Precautionary Principle* is optimal in contexts, close to those that arise when the tipping point is perfectly known, where actions are decreasing at the beginning of the dynamics. Under those circumstances, the decision-makers wants to start with high actions as the tipping point seems very far away, and decrease actions later. Capping actions is a way to control the decision-maker's incentives to choose those high actions earlier on.

Organization. Section 2 reviews the literature. Section 3 presents the model. Section 4 presents two benchmarks: the case where the rate of disaster follows a homogeneous Poisson process and the case where the tipping point is known. Section 5 introduces some uncertainty on the tipping point. Section 6 deals with the case of deep uncertainty, stressing the time-inconsistency problem that arises in this context and presenting the commitment solution. Section 7 analyzes the existence and properties of any *Stock-Markov Equilibrium*. Section 8 discusses the value of a *Precautionary Principle*. Section 9 briefly recaps our results and discusses possible extensions. Proofs are relegated into Appendices.

2. Literature Review

Irreversibility, Uncertainty and Information. Arrow and Fisher (1974), Henry (1974) and Freixas and Laffont (1984) were the first to show how a decision-maker should take more preventive stances when the consequences of irreversible choices are uncertain, the comparison being here with respect to the certainty case. Epstein (1980) has discussed general conditions under which this *Irreversibility Effect* prevails and proved that the value of waiting[17] increases when the decision-maker expects to benefit from a more informative signal (in the sense of Blackwell) on the future realizations of uncertainty. In those earlier models, information is exogenous while in many contexts in environmental economics, earlier actions also determine information structures. Hereafter, the probability of having passed the tipping point and possibly of learning it depends on the stock of past actions. Models with such endogenous information structures are scarce. Freixas and Laffont (1984) have studied a scenario in which more flexible actions increase the quality of future information, thus confirming the existence of the *Irreversibility Effect* while Miller and Lad (1984) have challenged this view in a model of conservation in which irreversible actions might also be more informative.[18] Salmi, Laiho and Murto (2019) study the trade-off faced by a decision-maker who must choose between acting now, which means taking a less informed decision but generating information useful in the sequel, and acting later, when being more informed. Only the speed of learning is endogenous.

[14]See Kydland and Prescott (1977), Persson and Tabellini (1994) for a nice survey of applications, Stockey (2002) for a more recent overview and Halac and Yared (2014) for recent developments.

[15]Epstein and O'Halloran (1999), Huber and Shipan (2002).

[16]See the literature on delegation in organisations as developed in Melumad and Shibano (1991), Alonso and Matousheck (2008), Martimort and Semenov (2008) and Amador and Bagwell (2013).

[17]Later coined as the *quasi-option value* by Graham-Tomasi (1995). See also Jones and Ostroy (1984) and Haneman (1989).

[18]Charlier (1997), Ramani, Richard and Trommetter (1992) and Ramani and Richard (1993) have also provided such models of endogenous information structures specializing their analysis to the context of *GMOs* and their development.

Greater actions accelerate the convergence of beliefs towards the true state.[19]

THE *Laissez-Faire* INTERPRETATION OF THE *Precautionary Principle*. Gollier, Jullien and Treich (2000) have built on the insights of the irreversibility literature to give some economic content to the *Precautionary Principle*. These authors interpret the *Precautionary Principle* as the incentives of a decision-maker to reduce his action below the level that would otherwise be optimal without uncertainty, when this action is taken before any information is learned. Much in the spirit of Kolstad (1996), Gollier, Jullien and Treich (2000) build a two-period model of pollution accumulation with exogenous information and draw conclusions on specific forms of utility functions that induce more precaution. Asano (2010) has focused on the comparison of optimal environmental policies without and with ambiguity, showing that the decision-maker's lack of confidence forces him to hasten the adoption of a policy, rather than postpone it. As we already pointed out, the decision-maker's behavior is optimal in these models and thus not constrained by a Constitutional *Precautionary Principle* in any way.[20] In other words, there would be no reason for drafting such legal principle in this setting. The *Laissez-Faire* solution suffices.

TIME-INCONSISTENCY. Our approach for characterizing a trajectory in a continuous time model with a time-inconsistency problem is similar to that developed in Karp and Lee (2003), Karp (2005, 2007), Ekeland, Karp and Sumaila (2015) and Ekeland and Lazrak (2006, 2008, 2010). These authors have analyzed various macroeconomic and growth models with time-inconsistency problems in continuous time. Beyond other differences in preferences, technology and dynamics of the evolution of the state variable, the source of such time-inconsistency in those models is the time-dependency of the discount factor. These models generalize the discrete-time models of Strotz (1955) and Laibson (1997)[21] who first introduced the possibility of time-inconsistent discounting. By contrast, in this paper time-inconsistency is generated by the fact that the expected probability of survival from any date on is stock-dependent. This property implies that the intertemporal rate of substitution between actions at two future dates decreases over time just as in most models with a time-dependent discount factor. A major difference is that the decision-maker in our context keeps some control over this time-dependency as it is affected by his own actions, while it is taken as given in the extant literature.

ON TIPPING POINTS. We build on a strand of the environmental economics literature which has focused on analyzing tipping points. Sims and Finoff (2016) have analyzed how irreversibility in environmental damage and irreversibility in sunk cost investment interact. Tsur and Zemel (1995) have investigated a problem of optimal resource extraction when extraction affects the probability that the resource becomes obsolete passed a certain threshold. Under deep uncertainty (unknown threshold) the initial state affects the optimal path and the decision-maker might end up exploiting the resource less than

[19]Taking a broader perspective, it is fair to recognize that the general framework proposed by the irreversibility literature has been applied to the economics of climate change with mixed success. Some authors have argued that this literature suggests that current abatements of greenhouse gaz emissions should be greater when more information will be available in the future (Chichilnisky and Heal, 1993; Beltratti, Chichilnisky and Heal, 1995; Kolstad, 1996; Gollier, Jullien and Treich, 2000; among others). Others like Ulph and Ulph (2012) have pointed out that the sufficient conditions given by Epstein (1980) for the *Irreversibility Effect* to hold may fail even in simple models of global warming.

[20]This feature is shared by other models in the field like Immordino (2000, 2005) and Gonzales (2008).

[21]See also Carrillo and Mariotti (2000), Harris and Laibson (2001) and O'Donoghue and Rabin (2003).

under certainty, maybe up to the point of stopping exploitation; an extreme form of precaution. In our model, the probability of the catastrophe is never zero once the activity has been started[22] and foregoing it is never optimal. In a model of optimal control of atmospheric pollution, Tsur and Zemel (1996) have shown how uncertainty on a tipping point introduces a multiplicity of possible equilibrium values. Finally, Liski and Salanié (2018) have also studied a model with unknown tipping points and deep uncertainty, but with different concerns. Their analysis focuses on the commitment scenario. Instead, we stress an important time-inconsistency problem in a context of deep uncertainty.

3. THE MODEL

TECHNOLOGY. A decision-maker (thereafter DM) runs a project which puts the environment at risk. Time is continuous. Let $r > 0$ be the discount rate. Let also $\mathbf{x} = (x(\tau))_{\tau \geq 0}$ (resp. $\mathbf{x}_t = (x(\tau))_{\tau \geq t}$) denote an actions plan (resp. the continuation of such a plan from date t on).

The project may induce a catastrophe, an event that follows a Poisson process with a (non-homogeneous) rate $\theta(t)$.[23] That rate depends on the stock $X(t) = \int_0^t x(\tau)d\tau$ of past actions that have already been taken before date t. More precisely, we postulate

$$\theta(t) = \theta_0 + \Delta \mathbb{1}_{\{X(t) > X_0\}} \tag{3.1}$$

where X_0 is a *tipping point*. Although it remains quite close to a homogeneous Poisson process, and indeed it is so before and after the tipping point, this specification features history-dependence on past actions. Indeed, when the cumulative stock of past actions $X(t)$ passes the *tipping point* X_0, the arrival rate jumps from θ_0 to $\theta_1 > \theta_0$. Let $\Delta = \theta_1 - \theta_0 > 0$ measure this jump.

PREFERENCES. Action $x(t)$ yields a surplus (net of the action cost) at date t worth $\zeta x(t) - \frac{x^2(t)}{2}$ where $\zeta > 0$. Action $x(t)$ belongs to an interval $\mathcal{X} = [0, 2\zeta]$ so that surplus remains non-negative under all circumstances below. Had he been myopic, DM would maximize his current payoff by choosing $x^m(t) = \zeta$ at any date $t \geq 0$. This myopic action is an important benchmark to assess the impact and origins of the precautionary motives that pertain to the different scenarios we investigate below.

If a disaster occurs at date t, DM incurs an irreversible flow of damages $-D$ from that date on. With a Poisson process, the long run probability of such event is one. The discounted welfare loss is thus $\frac{D}{r}$. Everything happens as if $\frac{D}{r}$ was paid upfront and DM would also enjoy D, viewed as the current benefit of not having a disaster, at any point in time as long as there is no disaster. To capture the detrimental and irreversible impact of a disaster, we also assume that, if such an event arises at date t, the flow surplus is no longer realized from that date on. A justification is that, the disaster is such a large event that production may no longer be possible afterwards. We will thus think of the benefit of not facing a disaster as the (not incurred) damage together with the surplus, and we accordingly define DM's such gain as:

$$u(x(t)) \equiv \zeta x(t) - \frac{x^2(t)}{2} + D.$$

[22] In fact, the probability of the catastrophe in the long run is one.

[23] The probability that a disaster arises over an interval $[t, t + dt]$ is thus $\theta(t)e^{-\int_0^t \theta(\tau)d\tau}dt$ and the probability that there has been no disaster up to date t is $e^{-\int_0^t \theta(\tau)d\tau}$.

WELFARE. Using the Poisson specification of the arrival rate of a disaster, DM's expected discounted welfare at date 0 if he adopts a plan $\mathbf{x} = (x(t))_{t\geq 0}$ can be expressed as:

$$\int_0^{+\infty} e^{-\left(rt+\int_0^t \theta(\tau)d\tau\right)} u(x(t))dt - \frac{D}{r}. \tag{3.2}$$

DM enjoys both the surplus and the flow benefit of not incurring a disaster as long as there is no such disaster up to date t, i.e., with probability $e^{-\int_0^t \theta(\tau)d\tau}$. Throughout the paper, we will specialize this expression to various informational environments.

4. BENCHMARKS

4.1. *No Irreversibility with Homogeneous Poisson*

We start with the simplest case where DM has no control over the arrival rate of a disaster, which is kept constant and equal to an exogenous parameter θ_0. This scenario corresponds to the case where the tipping point is at infinity, i.e., $X_0 = +\infty$. Specializing our previous formula (3.2), expected welfare can thus be written as:

$$\int_0^{+\infty} e^{-\lambda_0 t} u(x(t))dt - \frac{D}{r}$$

where, for future reference, we denote $\lambda_0 = r + \theta_0$ the effective discount rate that applies once the possibility of a disaster is taken into account.

Since he cannot influence the arrival rate of the disaster, DM always maximizes current surplus. The *myopic action* is always optimal:

$$x^m(t) = \zeta \quad \forall t \geq 0.$$

It naturally follows that the *Precautionary Principle* is irrelevant in this no-uncertainty setting. Any cap on actions would either be irrelevant or reduce welfare.

The net present value of this project is positive whenever

$$\frac{\lambda_1}{\lambda_0}\mathcal{V}_\infty \geq \frac{D}{r} \tag{4.1}$$

where, for future reference, we denote the myopic gain by $\lambda_1\mathcal{V}_\infty = u(\zeta) = \frac{\zeta^2}{2} + D$, and the effective discount rate that applies when the tipping point has been passed by $\lambda_1 = r + \theta_1$.

4.2. *Known Tipping Point*

Let define t_0 as the earliest date at which the tipping point is reached, namely:

$$t_0 = \min\left\{t \geq 0 \text{ s.t. } X(t) = X_0\right\}.$$

Had DM chosen to always act myopically, the tipping point would be reached at $t^m = \frac{X_0}{\zeta}$.

With these notations at hands, we may rewrite DM's expected welfare as:

$$\int_0^{t_0} e^{-\lambda_0 t} u(x(t))dt + e^{-\lambda_0 t_0} \int_{t_0}^{+\infty} e^{-\lambda_1 (t-t_0)} u(x(t))dt - \frac{D}{r}.$$

The first integral stems for welfare before the tipping point. This term is identical to that found for an homogeneous Poisson process, although now the upper bound of the interval is the date t_0 at which the tipping point is reached. The second integral stands for welfare after the tipping point, weighted by the probability of survival up to that date t_0. The only difference is that the arrival rate from that date on has now jumped up.

Dynamic Programming. Consider an action plan $\mathbf{x}_t = \{x(\tau)\}_{\tau\geq t}$ from date t onwards. If the stock at date t is X, this plan induces a stock process $\widetilde{X}(\tau; X, t)$ which evolves as:

$$\widetilde{X}(\tau; X, t) = X + \int_t^\tau x(s)ds. \tag{4.2}$$

After having passed the tipping point at date t_0, DM always chooses the myopic optimal action ζ and gets, from that date on, a discounted continuation payoff worth $\mathcal{V}_\infty$. For simplicity, we now omit the discounted damage $-\frac{D}{r}$ (although it has of course to be counted to assess the *ex ante* value of the project). We define the value function $\widetilde{\mathcal{V}}^{ck}(X, t; X_0)$, conditionally on having not yet faced a disaster, with a survival rate being $e^{-\theta_0 t}$ in this scenario where the value of the tipping point is known being at X_0, as

$$\widetilde{\mathcal{V}}^{ck}(X, t; X_0) \equiv \sup_{t_0, \mathbf{x}_t, \widetilde{X}(\cdot) \text{ s.t. } (4.2) \text{ and } \widetilde{X}(t_0; X, t) = X_0} \int_t^{t_0} e^{-\lambda_0(\tau - t)} u(x(\tau))d\tau + e^{-\lambda_0(t_0 - t)}\mathcal{V}_\infty.$$[24]

Below, we will sometimes slightly abuse notations and, for simplicity, write $\widetilde{X}(\tau; X) \equiv \widetilde{X}(\tau; X, 0)$, in which case the trajectory obeys to

$$\widetilde{X}(\tau; X) = X + \int_0^\tau x(s)ds. \tag{4.3}$$

Observe that we can write $\widetilde{\mathcal{V}}^{ck}(X, t; X_0) = \mathcal{V}^{ck}(X; X_0)$ for all $t \geq 0$, where the current value function $\mathcal{V}^{ck}(X; X_0)$ verifies

$$\mathcal{V}^{ck}(X; X_0) \equiv \sup_{\tau_0, \mathbf{x}, \widetilde{X}(\cdot) \text{ s.t. } (4.3) \text{ and } \widetilde{X}(\tau_0; X) = X_0} \int_0^{\tau_0} e^{-\lambda_0 \tau} u(x(\tau))d\tau + e^{-\lambda_0 \tau_0}\mathcal{V}_\infty. \tag{4.4}$$

We are now ready to characterize this value function and the associated feedback rule.

Proposition 1 *The function $\mathcal{V}^{ck}(X; X_0)$ is continuously differentiable on $[0, X_0)$, continuous on $[0, X_0]$ and it satisfies the following HBJ equation*

$$\dot{\mathcal{V}}^{ck}(X; X_0) = -\zeta + \sqrt{2\lambda_0 \mathcal{V}^{ck}(X; X_0) - 2D}, \quad \forall X < X_0. \tag{4.5}$$[25]

$\mathcal{V}^{ck}(X; X_0)$ is decreasing and strictly concave for $X \in [0, X_0)$ with the boundary condition

$$\mathcal{V}^{ck}(X_0; X_0) = \mathcal{V}_\infty. \tag{4.6}$$

[24] Observe that this expression of $\mathcal{V}(X, t)$ is valid both for $X < X_0$, and for $X \geq X_0$ provided that we use the convention $t_0 = t$ in that latter case.

[25] At $X = X_0$, this derivative is in fact a left-derivative but we use the same notation for simplicity.

The optimal feedback rule is such that

$$\sigma^{ck}(X;X_0) = \begin{cases} \zeta + \underbrace{\dot{\mathcal{V}}^{ck}(X;X_0)}_{\text{Irreversibility Effect}} & \text{for } X \in [0, X_0), \\ \zeta & \text{for } X \geq X_0. \end{cases} \tag{4.7}$$

Moreover, $\sigma^{ck}(X;X_0)$ is decreasing in X for $X \in [0, X_0)$.

Actions profile. The optimal action goes through two distinct phases. In the first *precautionary phase*, i.e., before reaching the tipping point, DM chooses an action which remains below the myopic optimum. The intuition is straightforward. Actions that have been taken in the past have a long-lasting impact since they may contribute to passing the tipping point earlier on. Reducing such actions keeps the probability that a disaster arises earlier at a low level. More precisely, the quantity $-\dot{\mathcal{V}}^{ck}(X;X_0)$ found on the right-hand side of (4.7) is in fact the Lagrange multiplier for the irreversibility constraint

$$\int_0^{t_0} x(\tau)d\tau = X_0 - X \tag{4.8}$$

where t_0 is here the date at which the tipping point X_0 is reached starting from an arbitrary level of the stock X. As X increases without having yet reached X_0, this irreversibility constraint becomes more demanding, and the value function is decreasing. Actions are reduced below the myopic optimum to account for this *Irreversibility Effect.*

The optimal action is decreasing over time before the tipping point is passed. All actions taken during the precautionary phase have the same marginal contribution to the overall stock. Because of discounting, DM prefers to choose the highest actions earlier on and the lowest ones when approaching the tipping point. Expressed in terms of the value function, this monotonicity boils down to the strict concavity of $\mathcal{V}^{ck}(X;X_0)$ during the precautionary phase. The value function becomes flat once the tipping point has been passed. By then, DM knows that his actions will no longer have any impact on the arrival rate of a disaster. We are back to the homogeneous case studied in Section 4.1. The optimal action is at the myopic optimum from that date on. When the tipping point is known, the optimal action path is necessarily non-monotonic with actions being first decreasing and then jumping up to the myopic outcome beyond the tipping point.

In summary, the *Precautionary Principle* is also irrelevant here. If the action is worth undertaking, capping actions early on would only transfer utility from the present to the future, with a net welfare loss.

Tipping Point. Because actions are now lower than the myopic optimum over the first phase, the tipping point is reached at a date $t_0 > t^m$. The intuition for this result is as follows. By pushing a bit further in the future the date at which the tipping point is reached by a small amount dt_0, DM incurs a welfare loss since, over the precautionary phase, the action is below the myopic optimum. DM is therefore getting less than the optimal surplus over a longer period of time. Taking into account discounting and the probability that no disaster has ever occurred before date t_0, this marginal loss can be expressed in terms of date 0 utils by discounting payoffs at a rate $\lambda_0 = r + \theta_0$ as:

$$\underbrace{e^{-\lambda_0 t_0}\left[\zeta x - \frac{x^2}{2}\right]_{\sigma^{ck}(X_0^-;X_0)}^{\zeta} dt_0 = \frac{e^{-\lambda_0 t_0}}{2}\left(\zeta - \sigma^{ck}(X_0^-;X_0)\right)^2.}_{\text{Marginal loss from not choosing the myopic action over } [t_0, t_0+dt_0]} \tag{4.9}$$

Moreover, by pushing a bit further in the future the date at which the tipping point is reached by a small amount dt_0, the feasibility constraint is hardened. It has a cost on DM's expected payoff which is worth (when expressed in current value)

$$-\frac{\partial \widetilde{X}}{\partial t}(t;X)|_{(t_0;X_0)}\dot{\mathcal{V}}^{ck}(X_0^-;X_0)e^{-\lambda_0 t_0}dt_0 = -x(t_0^-)\dot{\mathcal{V}}^{ck}(X_0^-;X_0)e^{-\lambda_0 t_0}dt_0. \tag{4.10}$$

Finally, pushing a bit further that date t_0 by dt_0 maintains the arrival rate of a disaster at its low level θ_0. By doing so, DM is less likely to losing not only the surplus $\frac{\zeta^2}{2}$ achieved with the myopic action that is optimal for $t \geq t_0$ but also the flow damage D in case a disaster occurs. Taking into account the discounted probability of a disaster from date t_0 on, the benefit (still expressed in terms of date 0 utils) of delaying the date at which the tipping point is reached by dt_0 can be written as:

$$\underbrace{\Delta u(\zeta)e^{-\lambda_0 t_0}\left(\int_{t_0}^{+\infty} e^{-\lambda_1(t-t_0)}dt\right)dt_0}_{\text{Marginal benefit of delaying the tipping point by } dt_0} \equiv \Delta\mathcal{V}_\infty e^{-\lambda_0 t_0}dt_0. \tag{4.11}$$

For future reference, we may thus define the net marginal benefit from pushing the tipping point further by dt_0 that is obtained when gathering (4.9), (4.10) and (4.11) above as

$$\left(\Delta\mathcal{V}_\infty - \frac{1}{2}\left(\sigma^{ck}(X_0^-;X_0) - \zeta\right)^2 + \sigma^{ck}(X_0^-;X_0)\dot{\mathcal{V}}^{ck}(X_0^-;X_0)\right)e^{-\lambda_0 t_0}dt_0$$

The optimal time t_0 at which the tipping point is reached (starting from an initial level of the stock which is nil) is obtained when this term is zero; a condition which becomes[26]

$$\zeta t_0 - X_0 = \left(\zeta - \sqrt{2\lambda_0\mathcal{V}_\infty - 2D}\right)\frac{1 - e^{-\lambda_0 t_0}}{\lambda_0}. \tag{4.12}$$ [27]

Bounds. The value function $\mathcal{V}^{ck}(X;X_0)$ remains above its long-term limit $\mathcal{V}_\infty$ reached when the tipping point has been passed. More interestingly, actions are always strictly positive even in the first precautionary phase (if the NPV of the project is positive).

Proposition 2 *$\mathcal{V}^{ck}(X;X_0)$ and $\sigma^{ck}(X;X_0)$ admit the following bounds*

$$\mathcal{V}_\infty \leq \mathcal{V}^{ck}(X;X_0) < \frac{\lambda_1}{\lambda_0}\mathcal{V}_\infty \quad \forall X, \tag{4.13}$$

[26]See the Appendix for details.

[27]It is worth pointing out that, for the constellation of parameters under consideration, passing the tipping point is always optimal. In other words, D is not so large as to make the project not valuable upfront or along the course of actions.

$$\sqrt{2\lambda_0\mathcal{V}_\infty - 2D} \leq \sigma^{ck}(X;X_0) \leq \zeta \quad \forall X.^{28} \tag{4.14}$$

Positive Net Present Value. The project has a positive net present value whenever

$$\mathcal{V}^{ck}(0,X_0) \geq \frac{D}{r}. \tag{4.15}$$

That the arrival rate of a disaster increases to θ_1 once the tipping point is passed, implies that expected welfare is lower than with an homogeneous Poisson process corresponding to a fixed arrival rate θ_0. The condition for running the technology is thus more stringent as it can be seen from comparing (4.1) and (4.15). In the sequel, we will ensure that the project always has a positive net present value by imposing the slightly stronger condition

$$\mathcal{V}_\infty \geq \frac{D}{r}. \tag{4.16}$$

5. UNCERTAINTY ON THE TIPPING POINT

Consider the more realistic case where the tipping point is not known at the time of starting the project. We also suppose that, DM knows when the tipping point is passed. The tipping point X is now a random variable drawn on $\left[0,\overline{X}\right]$[29] from a known (and atomless) distribution F. Let f be the corresponding (positive) density.

Dynamic Programming. Consider a process of the form (4.2) which is everywhere increasing and continuously differentiable. As times passes, the stock $\widetilde{X}(\tau;X,t)$ goes through different possible values of the tipping point. Formally, we may also describe this cumulative process by the time $\widetilde{T}(\widetilde{X};X,t) \geq t$ at which the stock reaches a level $\widetilde{X} \geq X$.

Accordingly, we also define the value function $\widetilde{\mathcal{V}}_1^u(X,t)$ (resp. $\widetilde{\mathcal{V}}_2^u(X,t)$) as DM's optimal intertemporal payoff starting from date t onwards when the stock level at date t is X given that DM knows (resp. ignores) that the tipping point has been passed, an event of probability $F(X)$, and there has been no catastrophe up to date t, an event of probability $e^{-\theta_0 t}$ if the tipping point has not been passed. Since the optimal action from date t on is the myopic optimum and the arrival rate of a disaster is θ_1, we have

$$\widetilde{\mathcal{V}}_1^u(X,t) = \lambda_1\mathcal{V}_\infty \int_t^{+\infty} e^{-r(\tau-t)}e^{-\theta_1(\tau-t)}d\tau = \mathcal{V}_\infty.$$

When the tipping point has not been passed yet, an event of probability $1-F(X)$, DM believes that the tipping point is actually drawn from a truncated distribution with density $\frac{f(\widetilde{X})}{1-F(X)}$ for $\tilde{X} \geq X$. On top, DM also knows that the probability of survival

[28]Condition (4.16) below, which ensures that the project has a positive NPV, also writes as $\mathcal{V}_\infty > \frac{1}{r}\left(\lambda_1\mathcal{V}_\infty - \frac{\zeta^2}{2}\right)$ or $\frac{\zeta^2}{2} > \lambda_1\mathcal{V}_\infty - r\mathcal{V}_\infty = \theta_1\mathcal{V}_\infty$ which implies $\frac{\zeta^2}{2} > \Delta\mathcal{V}_\infty$ and thus $\sqrt{2\lambda_0\mathcal{V}_\infty - 2D}$ exists.

[29]With the convention $\overline{X} = +\infty$ for an infinite support.

is $e^{-\theta_0 t}$. Taking into account discounting and the possibility of a regime shift at date $\widetilde{T}(\widetilde{X};X,t)$, we thus obtain the following expression for $\widetilde{\mathcal{V}}_2^u(X,t)$ conditionally on having not yet faced a disaster:

$$\widetilde{\mathcal{V}}_2^u(X,t) = \sup_{\mathbf{x}_t,\widetilde{X}(\cdot) \text{ s.t. } (4.2)} \frac{1}{1-F(X)} \int_X^{+\infty} \left(\int_t^{\widetilde{T}(\widetilde{X};X,t)} e^{-r(\tau-t)} e^{-\theta_0(\tau-t)} u(x(\tau)) d\tau \right.$$
$$\left. + e^{-\theta_0(\widetilde{T}(\widetilde{X};X,t)-t)} \int_{\widetilde{T}(\widetilde{X};X,t)}^{+\infty} e^{-r(\tau-t)} e^{-\theta_1(\tau-\widetilde{T}(\widetilde{X};X,t))} \lambda_1 \mathcal{V}_\infty d\tau \right) f(\widetilde{X}) d\widetilde{X}. \tag{5.1}$$

The difficulty here is that the maximand in (5.1) depends on both the action plan $\mathbf{x}_t$ and the inverse $\widetilde{T}(\widetilde{X};X,t)$ of the stock accumulation that this plan induces; a quite unusual feature. Next Lemma simplifies the optimization problem.

LEMMA 1 *The value function $\widetilde{\mathcal{V}}_2^u(X,t)$ satisfies $\widetilde{\mathcal{V}}_2^u(X,t) \equiv \mathcal{V}^u(X)$ for all (X,t) where $\mathcal{V}^u(X)$ is defined as:*

$$(1-F(X))\mathcal{V}^u(X) \equiv \sup_{\mathbf{x},\widetilde{X}(\cdot) \text{ s.t. } (4.3)} \mathcal{V}_\infty \int_0^{+\infty} e^{-\lambda_0 \tau} f(\widetilde{X}(\tau;X)) x(\tau) d\tau$$
$$+ \int_0^{+\infty} e^{-\lambda_0\tau} \left(1 - F(\widetilde{X}(\tau;X))\right) u(x(\tau)) d\tau.^{30} \tag{5.2}$$

In this scenario, DM always knows the arrival rate of a catastrophe. Indeed, the mere fact of knowing that the tipping point will be known when it is passed is enough to know this arrival rate, either after or before the tipping point. The state of the system can be reduced to the current stock of past actions exactly as when the tipping point is known.

HAMILTON-BELLMAN-JACOBI (HBJ) EQUATION. The maximization problem (5.2) has a recursive structure. As a consequence, the *Principle of Dynamic Programming* applies. Next proposition presents the HBJ equation satisfied by $\mathcal{V}^u(X)$, together with a characterization of the optimal feedback rule $\sigma^u(X)$.

PROPOSITION 3 *If the function $\mathcal{V}^u(X)$ is continuously differentiable, it satisfies the following HBJ equation*[31]

$$\dot{\mathcal{V}}^u(X) = \frac{f(X)}{1-F(X)} \left(\mathcal{V}^u(X) - \mathcal{V}_\infty\right) - \zeta + \sqrt{2\lambda_0 \mathcal{V}^u(X) - 2D} \tag{5.3}$$

[30] It is straightforward to check that the curent value function $\widetilde{\mathcal{V}}^u(X)$ is non-increasing in X and thus almost everywhere differentiable (see the Appendix for details). In the sequel, we will look for a value function that is actually continuously differentiable. From there, we will deduce a Hamilton-Bellman-Jacobi equation satisfied by this continuously differentiable value function. A *Verification Theorem* then provides sufficient conditions satisfied by the candidate solution.

[31] For what follows, it is important to remind the heuristic derivation of this HBJ equation. By the *Principle of Dynamic Programming,* the payoff $\mathcal{V}^u(X)$ is obtained by piecing together an optimal action path $\mathbf{x}_0^\varepsilon$ over an arbitrary interval $[0,\varepsilon]$ with a continuation path $\mathbf{x}_\varepsilon$ that yields the corresponding (non-discounted) continuation payoff $\mathcal{V}^u(\widetilde{X}(t+\varepsilon;X,t))$. The HBJ equation is then obtained by making the commitment period ϵ arbitrarily small, taking Taylor expansions while assuming that the function $\mathcal{V}^u(X)$ is continuously differentiable.[32] Reciprocally, Proposition 4 shows that a continuously differentiable solution to the HBJ equation satisfying the boundary condition (5.4) is the value function.

with the boundary condition

$$\lim_{X \to +\infty} (1 - F(X))(\mathcal{V}^u(X) - \mathcal{V}_\infty) = 0. \tag{5.4}$$

The optimal feedback rule is

$$\sigma^u(X) = \zeta + \underbrace{\dot{\mathcal{V}}^u(X)}_{\textit{Irreversibility Effect}} \underbrace{- \frac{f(X)}{1 - F(X)} (\mathcal{V}^u(X) - \mathcal{V}_\infty)}_{\textit{Learning Effect}}. \tag{5.5}$$

Even though there is uncertainty on where the tipping point lies, the optimal trajectory takes into account that passing the tipping point remains an irreversible act which leads to lower actions below the myopic outcome. Everything happens as if $\dot{\mathcal{V}}^u(X)$ was now an average Lagrange multiplier for all possible irreversibility constraints associated with future values of the tipping point. As in Section 4.2, this *Irreversibility Effect* calls for reducing actions below the myopic optimum.

Under uncertainty on the tipping point, DM also knows that, starting from a current stock X increasing action today by dx over a period of time of length dt makes it more likely that the tipping point will be passed in the very next future since the stock will increase by $dxdt$. The conditional probability of such increase would be $\frac{f(X)}{1-F(X)}dxdt$ while the expected welfare loss associated with learning that the tipping point has been passed would be

$$\frac{f(X)}{1 - F(X)} (\mathcal{V}^u(X) - \mathcal{V}_\infty)\, dxdt.$$

Reducing actions allows DM to avoid incurring this welfare loss. Because this loss changes as the probability of being near the tipping point evolves, we call it a *Learning Effect*.

Last, the *Precautionary Principle* is irrelevant in this setting. This is because reducing actions earlier on does not directly influence the rate of arrival of catastrophes. It would reduce welfare by transferring utility from the present to the future, postponing the tipping point over time, but resulting in a welfare loss.

Proposition 4 *Suppose that F has infinite support. There exists a unique continuously differentiable function, the current value function, $\mathcal{V}^u(X)$ satisfying the HBJ equation (5.3) and the boundary condition (5.4). $\mathcal{V}^u(X)$ and $\sigma^u(X)$ admit the following bounds*

$$\mathcal{V}_\infty < \mathcal{V}^u(X) < \frac{\lambda_1}{\lambda_0} \mathcal{V}_\infty \quad \forall X \geq 0, \tag{5.6}$$

$$\sqrt{2\lambda_0 \mathcal{V}_\infty - 2D} < \sigma^u(X) < \zeta \quad \forall X \geq 0. \tag{5.7}$$

From (5.1), $\mathcal{V}_\infty$ is in fact a lower bound for the value function $\mathcal{V}^u(X)$. It is readily obtained by following a sub-optimal strategy consisting in always adopting the myopic action under all circumstances.

Condition (5.7) below also shows that *not acting* is never optimal.[33] The example of an exponential distribution that is developed below provides more intuition on those bounds on actions. Actions come close to the lower bound when it is very likely that the tipping point will soon be passed. The *Irreversibility Effect* is at its maximum. Any further action may then have a huge impact. Actions are closer to the myopic outcome when instead they have little impact on the probability of having passed the tipping point.

Exponential Distributions. Here, closed-form solutions are readily obtained. The optimal action is stationary, always positive and independent of the current stock while the stock evolves linearly over time.

Proposition 5 *Suppose that X is exponentially distributed over $\mathbb{R}_+$, i.e., $f(X) = ke^{-kX}$ and $F(X) = 1 - e^{-kX}$ for some $k > 0$. Closed forms for the current value function, the optimal feedback rule and the optimal stock are respectively obtained as*

$$\mathcal{V}^u(X) = \mathcal{V}_\infty + \frac{\lambda_0}{k^2} + \frac{\zeta}{k} - \sqrt{\left(\frac{\lambda_0}{k^2} + \frac{\zeta}{k}\right)^2 - 2\frac{\Delta\mathcal{V}_\infty}{k^2}} \quad \forall X \geq 0, \tag{5.8}$$

$$\sigma^u(X) = \sqrt{\left(\frac{\lambda_0}{k} + \zeta\right)^2 - 2\Delta\mathcal{V}_\infty} - \frac{\lambda_0}{k} > 0 \quad \forall X \geq 0, \tag{5.9}$$

$$X^u(t) = \left(\sqrt{\left(\frac{\lambda_0}{k} + \zeta\right)^2 - 2\Delta\mathcal{V}_\infty} - \frac{\lambda_0}{k}\right) t \quad \forall t \geq 0. \tag{5.10}$$

Those expressions provide important insights on how uncertainty shapes optimal trajectories. By varying the parameter k, we may go from the pure uninformative Laplacian distribution over the positive real line ($k \to 0$) to the Dirac distribution putting mass one at zero ($k \to +\infty$); meaning the tipping point is passed almost immediately. Moving towards the Laplacian world ($k \to 0$) can admittedly be viewed as a metaphor for a context where DM is agnostic on where tipping points lies. The feedback $\sigma^u(X)$ then converges towards the myopic optimum $x^m = \zeta$. Indeed, with such large ignorance on where the tipping point lies, the probability that the tipping point has been passed remains always the same at any point in time, namely almost zero. In other words, actions that have already been taken have no impact on the probability of having passed the tipping point and DM is as well off always opting for the myopic action.

When the distribution comes closer to a Dirac distribution at zero ($k \to +\infty$), the feedback rule $\sigma^u(X)$ converges towards the lower bound $\sqrt{2\lambda_0\mathcal{V}_\infty - 2D}$. Intuitively, DM refrains from taking large actions because he expects that, otherwise, the stock quickly crosses almost all values of the tipping point, increasing the likelihood of a disaster.

Positive Net Present Value. DM now chooses to run the risky technology when:

$$\mathcal{V}_2^u(0,0) = \mathcal{V}^u(0) \geq \frac{D}{r}.$$

[33] Observe also that having an increasing stock process, as requested to ensure that the smooth stock profile is invertible, requires $\sigma^u(X) > 0$, a condition that is implied by Condition (5.7).

Because $\mathcal{V}^u(0) \geq \mathcal{V}_\infty$, this condition is implied by (4.16).

Long-Run Behavior. Beyond the exponential scenario, it is interesting to describe the long-run behavior of the solution more generally. To this end, we first define the function $R(Y) = f(F^{-1}(1-Y))$ for all $Y \in [0,1]$ and assume that $\lim_{X\to+\infty} -\frac{f'(X)}{f(X)}$ exists and is positive. Let $\dot{R}(0) > 0$ denote this limit. This parameter plays the same role as k in the case of an exponential distribution. The asymptotic behavior of the solution is characterized below.

Proposition 6 *$\mathcal{V}^u(X)$ and $\sigma^u(X)$ admit the following approximations when X is large:*

$$\mathcal{V}^u(X) - \mathcal{V}_\infty \sim_{X\to+\infty} \frac{\lambda_0}{\dot{R}(0)^2} + \frac{\zeta}{\dot{R}(0)} - \sqrt{\left(\frac{\lambda_0}{\dot{R}(0)^2} + \frac{\zeta}{\dot{R}(0)}\right)^2 - \frac{2\Delta\mathcal{V}_\infty}{\dot{R}(0)^2}}.\text{[34]} \tag{5.11}$$

$$\sigma^u(X) \sim_{X\to+\infty} \sqrt{\left(\frac{\lambda_0}{\dot{R}(0)} + \zeta\right)^2 - 2\Delta\mathcal{V}_\infty} - \frac{\lambda_0}{\dot{R}(0)} < \zeta. \tag{5.12}$$

6. DEEP UNCERTAINTY: TIME-INCONSISTENCY

Suppose now that DM does not even know whether the tipping point has been passed or not; a scenario thereafter coined as being one of deep uncertainty. The key difference with the less extreme scenario investigated in Section 5 is that DM can no longer switch to the myopic optimum once the tipping point has been passed since he ignores this event. Yet, DM must account for that possibility when choosing his action plan.

Survival Rate. We first compute DM's beliefs that a disaster will occur over an interval $[t, t+dt]$ if, starting from an initial stock $X \geq 0$ at date 0, the action plan $\bar{\mathbf{x}}^t$ is followed up to date t without incurring any disaster. Let $\widetilde{X}(t;X) = X + \int_0^t \bar{x}(\tau)d\tau$ be the corresponding stock and $\widetilde{T}(\widetilde{X};X)$ its inverse function. DM believes that the probability that a disaster will occur over the interval $[t, t+dt]$ is $g(t;\bar{\mathbf{x}}^t, X)dt$ where

$$g(t;\bar{\mathbf{x}}^t, X) = (1 - F(\widetilde{X}(t;X))\theta_0 e^{-\theta_0 t} + \int_0^{\widetilde{X}(t;X)} \theta_1 e^{-(\theta_0\widetilde{T}(\widetilde{X};X)+\theta_1(t-\widetilde{T}(\widetilde{X};X)))} f(\widetilde{X})d\widetilde{X}.$$

This expression takes into account that, for any date $t \geq 0$, all tipping points $\widetilde{X}$ such that $\widetilde{X} \leq \widetilde{X}(t;X)$ have already been passed and the arrival rate of a disaster has thus increased from θ_0 to θ_1. If instead the tipping point is at $\widetilde{X}$ such that $\widetilde{X} > \widetilde{X}(t;X) \geq X$, the arrival rate remains θ_0. Let also $1 - G(t;\bar{\mathbf{x}}^t, X) = 1 - \int_0^t g(\tau;\bar{\mathbf{x}}^\tau, X)d\tau$ denote the probability of survival till date t if the path $\bar{\mathbf{x}}^t$ has been followed to that date.

[34] To illustrate, suppose that X is drawn according to the logistic distribution with density $f(X) = \frac{ke^{-k(X-X_0)}}{\left(1+e^{-k(X-X_0)}\right)^2}$. (Admittedly, this density is defined over the whole real line but negative values have a very low probability when k goes to $+\infty$.) As k increases towards $+\infty$, this distribution shifts more mass around the threshold X_0 so as to come closer to the scenario where the tipping point is known to be at that point. Yet, since $\dot{R}(0) = \lim_{X\to+\infty} -\frac{f'(X)}{f(X)} = k$, the optimal action, conditional on not having passed the tipping point, converges again towards the lower bound $\sqrt{2\lambda_0\mathcal{V}_\infty - 2D}$.

LEMMA 2 *When an increasing path $\widetilde{X}(t;X)$ has been followed up to date t, the probability of survival is*

$$1 - G(t;\bar{\mathbf{x}}^t, X) = e^{-\theta_0 t}\left(1 - \Delta e^{-\Delta t}\int_0^t F(\widetilde{X}(\tau;X))e^{\Delta\tau}d\tau\right). \tag{6.1}$$

A consequence of (6.1) and the fact that $F(X) \leq F(\widetilde{X}(t;X)) \leq 1$ for all $t \geq 0$ is that

$$e^{-\theta_1 t} \leq 1 - G(t;\bar{\mathbf{x}}^t, X) \leq (1 - F(X))e^{-\theta_0 t} + F(X)e^{-\theta_1 t} \quad \forall (X,t). \tag{6.2}$$

The right-hand side is the expected probability that no disaster has happened up to date t if DM takes a naive view and considers that only the initial stock matters to assess this probability. The left-hand side is the "worst" scenario where the tipping point has already been passed. It can also be interpreted as the long-term belief about the probability of survival. More generally, (6.1) shows how beliefs evolve along the trajectory. At the beginning, $\widetilde{X}(t;X)$ is close to X and the likelihood of having passed the tipping point close to $F(X)$. DM still believes that the arrival rate of a disaster is close to θ_0. As $\widetilde{X}(t;X)$ increases, it becomes more likely that the tipping point has been passed and this rate is thought being close to θ_1. Of course, the shape of the distribution function F matters to evaluate such beliefs. As F puts more mass around X, it becomes more likely that the tipping point has been passed early on and DM is more inclined to think that the arrival rate has already shifted to θ_1. Instead, if F puts more mass on higher values of X, DM believes that this rate remains θ_0 for a longer period of time.

VALUE FUNCTION. Consider a past history of actions $\bar{\mathbf{x}}^t$ with no disaster up to date t and a stock at date t given by $X = \int_0^t \bar{x}(s)ds$. From that date on, this stock will evolve as $\widetilde{X}(\tau;X,t) = X + \int_t^\tau x(s)ds$ where $\mathbf{x}_t = (x(s))_{s\geq t}$ is the stream of future actions. We define the value function (again gross of $-\frac{D}{r}$) as DM's continuation payoff starting from date t onwards given the past history. Importantly, the only information available to DM at date t is precisely the fact that, with probability $F(X)$, the tipping point has already been passed and no disaster has yet happened. Only the current stock X at date t thus matters to evaluate this continuation payoff. We thus denote this payoff as $\widetilde{\mathcal{V}}^c(X,t)$ where

$$\widetilde{\mathcal{V}}^c(X,t) \equiv \sup_{\mathbf{x}_t,\widetilde{X}(\cdot)\text{ s.t. }(4.2)} \int_0^X \left(\int_t^{+\infty} e^{-r(\tau-t)}e^{-\theta_1(\tau-t)}u(x(\tau))d\tau\right)f(\widetilde{X})d\widetilde{X} \tag{6.3}$$
$$+\int_X^{+\infty}\left(\int_t^{\widetilde{T}(\widetilde{X};X,t)} e^{-r(\tau-t)}e^{-\theta_0(\tau-t)}u(x(\tau))d\tau\right.$$
$$\left.+e^{-\theta_0(\widetilde{T}(\widetilde{X};X,t)-t)}\int_{\widetilde{T}(\widetilde{X};X,t)}^{+\infty} e^{-r(\tau-t)}e^{-\theta_1(\tau-\widetilde{T}(\widetilde{X};X,t))}u(x(\tau))d\tau\right)f(\widetilde{X})d\widetilde{X}.$$

Observe that $\widetilde{\mathcal{V}}^c(X,t)$ is a double integral, taken first over all possible values of the tipping point and second over time. Adding up the probabilities of all potential scenarios with no disaster up to date τ ($\tau \geq t$) amounts to re-organizing this double integral; first along time and second along possible values of the tipping point that have already been passed up to that time. Counting paths this way shows that this overall probability of survival is precisely $1 - G(\tau;\bar{\mathbf{x}}^\tau, X)$. Date τ-payoff is thus discounted not only with the interest rate but with this survival probability. This is a key difference with the scenarios described above where a cap on actions could only transfer utility across time. In the deep uncertainty scenario, it also becomes crucial to think about how to transfer utility across states. The summary statistic of this feature is the discount rate which is no longer constant but now depends on the trajectory. Next Lemma summarizes these findings.

LEMMA 3 *The value function* $\widetilde{\mathcal{V}}^c(X,t) \equiv \mathcal{V}^c(X)$ *satisfies*

$$\mathcal{V}^c(X) = \sup_{\mathbf{x}, \widetilde{X}(\cdot), \widetilde{Z}(\cdot) \text{ s.t. } (4.3)} \int_0^{+\infty} e^{-\lambda_0 \tau} \widetilde{Z}(\tau; \mathbf{x}^\tau, X) u(x(\tau)) d\tau \tag{6.4}$$

where $\widetilde{Z}(\tau; \mathbf{x}^\tau, X)$ *is the survival index defined as*

$$\widetilde{Z}(\tau; \mathbf{x}^\tau, X) = (1 - G(\tau; \mathbf{x}^\tau, X)) e^{\theta_0 \tau} = 1 - \Delta e^{-\Delta \tau} \int_0^\tau F(\widetilde{X}(s; X)) e^{\Delta s} ds. \tag{6.5}$$

The value function $\widetilde{\mathcal{V}}^c(X)$ depends on the current stock X through the impact that the future trajectory $\widetilde{X}(\tau; X)$ (for $\tau \geq 0$) has on the *survival index* $\widetilde{Z}(\tau; \mathbf{x}^\tau, X)$. That index measures how the probability of survival $1 - G(\tau; \bar{\mathbf{x}}^\tau, X)$ will evolve along the future trajectory relatively to the homogeneous case where its value would be $e^{\theta_0 \tau}$. Everything thus happens as if date τ-payoff is discounted at a path-dependent rate worth

$$\lambda_0 - \frac{\frac{\partial \widetilde{Z}}{\partial \tau}(\tau; \mathbf{x}^\tau, X)}{\widetilde{Z}(\tau; \mathbf{x}^\tau, X)} = \lambda_0 - \frac{\Delta(1 - F(\widetilde{X}(\tau; X)) - \widetilde{Z}(\tau; \mathbf{x}^\tau, X))}{\widetilde{Z}(\tau; \mathbf{x}^\tau, X)} = \lambda_1 - \frac{\Delta(1 - F(\widetilde{X}(\tau; X)))}{\widetilde{Z}(\tau; \mathbf{x}^\tau, X)}.$$

As the stock $\widetilde{X}(\tau; X)$ increases and more potential values of the tipping points have been passed, this term converges towards λ_1. Future payoffs are thus viewed as being less valuable than earlier ones and the less so as $\widetilde{X}(\tau; X)$ will have increased. Next Lemma confirms this fundamental property of such non-constant discounting.

LEMMA 4 *Fix a path of actions* $\mathbf{x}$ *from date 0 on, with a stock evolving as* $\widetilde{X}(\cdot; 0)$. *Consider three dates* $0 < t < \tau < \tau'$. *The following inequality holds:*

$$\frac{1 - G(\tau | \mathbf{x}^\tau, 0)}{1 - G(\tau' | \mathbf{x}^{\tau'}, 0)} < \frac{1 - G(\tau - t | \mathbf{x}^{\tau - t}, \widetilde{X}(t; 0))}{1 - G(\tau' - t | \mathbf{x}^{\tau' - t}, \widetilde{X}(t; 0))}. \tag{6.6}$$

Lemma 4 shows that the marginal rate of substitution between actions taken at two dates τ and τ' increases with t. In other words, DM values relatively more an action at a nearby date τ rather than an action taken at a later date $\tau' > \tau$ when this assessment is postponed. As time passes, a closer date τ-action is viewed as being relatively more attractive than a more far away date τ'-action. Intuitively, the right-hand side of (6.6) measures the extent to which surviving the same additional amount of time $\tau' - \tau$ is seen as being more unlikely as time passes and no disaster has yet happened, leading the DM to become more impatient. This monotonicity points at to a fundamental time-inconsistency in a context of deep uncertainty. Indeed, a solution for (6.4) for $X = 0$ corresponds to an action plan $\mathbf{x}^c(0)$ such that $x^c(\tau; 0)$ is not immune to re-optimization of the action plan when the stock has reached $X = \int_0^t x^c(s; 0) ds$ for $0 < t < \tau$.

COMMITMENT SOLUTION. Because of this time-inconsistency problem, dynamic programming techniques are not directly applicable to characterize the solution to the maximization problem (6.4). We thus rely on a more direct method by means of optimal control and Hamiltonian techniques. The main features are summarized in the next proposition.

Proposition 7 *There exists a solution to the optimization problem (6.4). The optimal path of actions and stock trajectory $(x^c(t;X), \widetilde{X}^c(t;X))$ satisfy the following necessary condition:*

$$x^c(t;X) = \zeta - \frac{\Delta}{\widetilde{Z}^c(t;X)} \int_0^{+\infty} f(\widetilde{X}^c(t+\tau;X))e^{\Delta\tau}\left(\int_\tau^{+\infty} e^{-\lambda_1 s}u(x^c(t+s;X))ds\right)d\tau \tag{6.7}$$

where

$$\widetilde{Z}^c(t;X) = 1 - \Delta e^{-\Delta t}\int_0^t F(\widetilde{X}^c(\tau;X))e^{\Delta\tau}d\tau.$$

The optimal action $x^c(t;X)$ results from a trade-off between increasing current payoff when moving towards the myopic optimum and making it more likely to pass the tipping point at that date; a familiar *Irreversibility Effect*. Indeed, observe that the marginal discounted benefits of increasing by dx the action $x^c(t;X)$ over an interval $[t, t+dt]$ is

$$e^{-\lambda_0 t}\widetilde{Z}^c(t;X)(\zeta - x^c(t;X))dxdt. \tag{6.8}$$

Such a marginal change of action modifies the whole future trajectory which is shifted upwards to $\widetilde{X}^c(t+\tau;X) + dxdt$ for $\tau \geq 0$. When evaluated from date 0 viewpoint, the cost of passing the tipping point at a date $\tau \geq t$, thereby increasing the arrival rate of accident by Δ and losing future surplus computed along the committed plan of actions from that date on is thus

$$\Delta e^{-\lambda_0 t}\left(\int_0^{+\infty} f(\widetilde{X}^c(t+\tau;X))e^{\Delta\tau}\left(\int_\tau^{+\infty} e^{-\lambda_1 s}u(x^c(t+s;X))ds\right)d\tau\right)dxdt. \tag{6.9}$$

The optimal action $x^c(t;X)$ balances (6.8) and (6.9).

For ease of comparison with our findings in Section 7, it is useful to rewrite the optimality condition at $t=0$ as

$$x^c(0;X) = \zeta + \dot{\mathcal{V}}^c(X) \tag{6.10}$$

where $\dot{\mathcal{V}}^c(X)$ again measures the overall shadow cost of increasing the current stock.[35]

Positive Net Present Value. If able to commit to an actions plan from date 0 on, DM can achieve a payoff $\mathcal{V}^c(0)$. The project has thus a positive net present value when

$$\mathcal{V}^c(0) \geq \frac{D}{r}.$$

From (6.2) and (6.4), it immediately follows that

$$\mathcal{V}^c(0) \geq \sup_{\mathbf{x}} \int_0^{+\infty} e^{-\lambda_1\tau}u(x(\tau))d\tau = \mathcal{V}_\infty.$$

Therefore, Condition (4.16) is again sufficient to ensure a positive net present value in a context of deep uncertainty.

[35]See the derivation of Equation (6.10) in Appendix C.

Time-Inconsistency. If the solution was time-consistent, the optimal action profile decided at date 0, namely $\mathbf{x^c}(X)$, would remain optimal at a future date $t > 0$ when the stock will have reached $\widetilde{X}^c(t;X)$. Formally, this would mean that

$$x^c(t';X) \equiv x^c(t'-t;\widetilde{X}^c(t;X)) \quad \forall X, \forall t' \geq t \geq 0. \tag{6.11}$$

Importantly, this condition never holds as shown below. The intuition is straightforward and follows our earlier insights. When date t comes, DM views increasing current action as more attractive than what was initially thought at date 0.

Proposition 8 *The action plan* $\mathbf{x}^c(X)$ *is not time-consistent.*

If time-consistency requirement (6.11) were to hold, we would also have

$$x^c(t;X) = x^c(0;\widetilde{X}^c(t;X)) \quad \forall X, \forall t \geq 0.$$

In other words, the optimal action at date t would only depend on the existing stock at that date. The fact that the optimal solution is not time-consistent thus also shows that a simple feedback rule that would only depend on the current stock X cannot be used to achieve the commitment payoff $\mathcal{V}^c(0)$. We will come back to this issue in Section 7 below. There, we consider feed-back rules that depend on X only, just as a time-consistent plan would require if one existed. That restriction can thus be viewed as being a meaningful one in a context where no such plan exists.[36]

7. DEEP UNCERTAINTY: *PSEUDO*-VALUE FUNCTION AND STOCK-MARKOV EQUILIBRIUM

By adopting a so called *Stock-Markov* feedback rule, DM chooses the same action $\sigma^*(X)$ at any given level of stock X irrespectively of how the past history led to that stock level. Of course, at an equilibrium of this sort, DM should stick to the feedback rule $\sigma^*(X)$ today because he expects to always abide to this rule in the future. Along such a *Stock-Markov* trajectory, the stock thus evolves according to

$$X^*(\tau;X) = X + \int_0^\tau \sigma^*(X^*(s;X))ds. \tag{7.1}$$

In the same vein, the survival index evolves in a way that is consistent with the *Stock-Markov* feedback rule $\sigma^*(X)$. Adapting (6.5), at any date τ and for any initial value of the stock X and along such a *Stock-Markov* trajectory, this index should satisfy

$$\mathcal{Z}^*(\tau;X) = 1 - \Delta e^{-\Delta\tau} \int_0^\tau F(X^*(s;X))e^{\Delta s}ds. \tag{7.2}$$

We may also adapt (6.4) and now define the *pseudo-value function* $\mathcal{V}^*(X)$ as DM's payoff function along such a *Stock-Markov* trajectory, namely

$$\mathcal{V}^*(X) = \int_0^{+\infty} e^{-\lambda_0\tau}\mathcal{Z}^*(\tau;X)u(\sigma^*(X^*(\tau;X)))d\tau. \tag{7.3}$$

[36]The expression of $\mathcal{V}^c(X)$ given in (6.4) suggests that the state of the system is best described by adding to the value of the current stock X another state variable that reflects how the probability of survival evolves in the future. Two trajectories that reach the same value for the current stock at date t and keep the same survival rate should be optimally continued the same way. Instead, two trajectories that have reached the same stock of past actions but are thought to survive with different probabilities might be pursued along two different paths. To expand state variables and restore the force of dynamic programming, we show in Appendix C how to use the *survival index* as another state variable.

Impulse Deviations. To express the equilibrium requirement that sticking to the feedback rule $\sigma^*(X)$ is optimal at any point along the trajectory, we follow an approach which is similar in spirit although different in details to that developed in Karp and Lee (2003), Karp (2005, 2007), Ekeland, Karp and Sumaila (2015) and Ekeland and Lazrak (2006, 2008, 2010). These authors have analyzed various macroeconomic and growth models with time-inconsistency problems. Roughly speaking it consists in importing the notion of perfect Markov equilibrium, familiar in discrete-time models, to a continuous time setting.The idea is to look at the benefits of deviating from the feedback rule for periods of commitment of arbitrarily small length; deriving from there conditions for the sub-optimality of such deviations.[37]

To this end, consider a possible deviation that would consist in committing to an action x for a period of length ε, reaching a stock level $X+x\varepsilon$, before jumping back to the above feedback rule σ^*. For such a deviation, actions evolve according to

$$y(x,\varepsilon,\tau;X)=\begin{cases} x & \text{if } \tau\in[0,\varepsilon],\\ \sigma^*(\widetilde{X}(x,\varepsilon,\tau;X)) & \text{if } \tau>\varepsilon \end{cases} \tag{7.4}$$

while the whole stock trajectory is modified as

$$\widetilde{X}(x,\varepsilon,\tau;X)=\begin{cases} X+x\tau & \text{if } \tau\in[0,\varepsilon],\\ X+x\varepsilon+\int_\varepsilon^\tau \sigma^*(\widetilde{X}(x,\varepsilon,s;X))ds & \text{if } \tau\geq\varepsilon. \end{cases} \tag{7.5}$$

By adopting the deviation (7.4)-(7.5), the survival index would also change as

$$\mathcal{Z}(x,\varepsilon,\tau;X)=1-\Delta e^{-\Delta\tau}\int_0^\tau F(\widetilde{X}(x,\varepsilon,s;X))e^{\Delta s}ds. \tag{7.6}$$

From this, we may define DM's deviation payoff $\mathcal{V}(x,\varepsilon;X)$ as

$$\mathcal{V}(x,\varepsilon;X)=\int_0^{+\infty} e^{-\lambda_0\tau}\mathcal{Z}(x,\varepsilon,\tau;X)u(y(x,\varepsilon,\tau;X))d\tau. \tag{7.7}$$

When ε is made arbitrarily small, we will refer to such deviations as *impulse deviations.*

We define a *Stock-Markov Equilibrium* $(\mathcal{V}^*(X),\sigma^*(X))$ i.e., a payoff function and a *Stock-Markov* feedback rule that are immune to such impulse deviations.

Definition 1 *$(\mathcal{V}^*(X),\sigma^*(X))$ is a Stock-Markov Equilibrium if $\mathcal{V}^*(X)$ as defined by (7.3) cannot be improved upon by any impulse deviation of the form (7.4)-(7.5) for ε made arbitrarily small, i.e.,*

$$\mathcal{V}^*(X)=\lim_{\varepsilon\to 0^+}\max_{x\in\mathcal{X}}\mathcal{V}(x,\varepsilon;X). \tag{7.8}$$

[37] To figure out how it could be done more formally, consider a discrete version of our model where DM would thus commit to an action over each period $[t,t+\varepsilon]$, $[t+\varepsilon,t+2\varepsilon]$, $...[t+n\varepsilon,t+(n+1)\varepsilon]$ (with $n\in\mathbb{N}$). It is then natural to focus on stationary Markov-perfect subgame equilibria for such a discrete game. In such an equilibrium, DM follows a feedback rule $\sigma^*_\varepsilon(X)$ that defines his current action in terms of the existing stock only. Of course, the equilibrium requirement imposes that this feedback rule is a best-response given DM's anticipations of his own future actions, which should themselves follow the same feedback rule although, of course, the stock at future dates has evolved according to past actions.

A FUNCTIONAL EQUATION SATISFIED BY $\mathcal{V}^*(X)$. Writing the equilibrium condition suggested by Definition 1 gives us some important properties.

PROPOSITION 9 *A continuously differentiable pseudo-value function $\mathcal{V}^*(X)$ satisfies the following functional equation:*

$$\dot{\mathcal{V}}^*(X) = -\zeta + \sqrt{2\lambda_0 \mathcal{V}^*(X) - 2D + 2\Delta F(X)\varphi(X)} \tag{7.9}$$

with the boundary condition

$$\lim_{X \to +\infty} \mathcal{V}^*(X) = \mathcal{V}_\infty \tag{7.10}$$

and where

$$\varphi(X) = \int_0^{+\infty} e^{-\lambda_1 \tau} u(\sigma^*(X^*(\tau; X)))d\tau. \tag{7.11}$$

The equilibrium feedback rule $\sigma^*(X)$ satisfies

$$\sigma^*(X) = \zeta + \dot{\mathcal{V}}^*(X). \tag{7.12}$$

Once envisioning the costs and benefits of an impulse deviation of the form (7.4)-(7.5), DM takes as given the fact that, in the future, he will stick to the equilibrium feedback rule and the stock will evolve accordingly. The future evolution of this stock is thus taken as given to assess the costs and benefits of any putative impulse deviation. Once a given stock level X has been reached, the marginal gain of increasing by an amount dx the current action $\sigma^*(X)$ over an interval of length ε small enough is approximately worth

$$\mathcal{Z}(\varepsilon, 0; X)u'(\sigma^*(X))\varepsilon dx = (\zeta - \sigma^*(X))\varepsilon dx$$

where the right-hand side follows from $\mathcal{Z}(\varepsilon, 0; X) = 1$. Such a change in the current action also modifies the whole future trajectory. The continuation payoff should now be evaluated at a higher stock, namely $X + (\sigma^*(X) + dx)dt$ instead of $X + \sigma^*(X)dt$. The marginal cost of increasing current action is thus

$$\dot{\mathcal{V}}^*(X)\varepsilon dx.$$

As in the scenario of Section 5 where it is known when the tipping point has been passed, the choice of an action at a given point in time again balances two effects. First, moving closer to the myopic optimum for a small period of time, increases current payoff. Second, increasing the current action raises future stock and thus accordingly reduces DM's continuation payoff since $\dot{\mathcal{V}}^*(X) < 0$. Indeed, for all possible values of the tipping points that lie above the current value of the stock, the corresponding irreversibility constraint is thereby hardened. Averaging over all possible such values of the tipping point again highlights a familiar *Irreversibility Effect.* This effect calls for reducing current action. The feedback rule (7.12) is precisely obtained when the marginal benefit of an impulse deviation is equal to its long run marginal cost.

The feedback rule (7.12) replicates (6.10). While the latter applies only at the start of the optimization period, the former applies everywhere; which captures the idea that each self of the decision-maker behaves as if he was jumpstarting a new optimization of the trajectory at any point in time.

EXISTENCE. At first glance, the functional equation (7.9) looks like a HBJ differential equation, although it is strikingly different. Indeed, it is *non-local* and forward-looking. It depends not only on the current stock but also on future values of the stock along the equilibrium trajectory. Once contemplating a deviation over an interval of an arbitrarily small length, DM takes as given the fact that, in the future, he will stick to the time-consistent feedback rule. The whole profile of these future actions which, from optimality of the feedback rule, depends on future values of the marginal current-value function, is thus taken as given to assess the cost and benefit of any putative deviation.

Characterizing the solution to the functional equation (7.9) together with the boundary condition (7.10) is a difficult endeavor because of the non-local nature of the solution which makes standard local techniques of no help. To make progresses, we first transform this functional equation into a system of differential equations which are respectively satisfied by the *pseudo*-value function $\mathcal{V}^*(X)$, namely (7.9) and the externality component of the payoff $\varphi(X)$, namely

$$(7.13) \quad \dot{\varphi}(X) = \frac{\lambda_1(\varphi(X) - \mathcal{V}_\infty) + \frac{1}{2}(\dot{\mathcal{V}}^*(X))^2}{\dot{\mathcal{V}}^*(X) + \zeta}.$$ [38]

The properties of this system are then easily analyzed under two different scenarios. First, when the distribution F has finite support (Proposition 10) and, second, with an unbounded support (Proposition 11). With a finite support, the boundary condition (7.10) requires that $\sigma^*(X) = \zeta$ and $\mathcal{V}^*(X) = \varphi(X) = \mathcal{V}_\infty$ for all $X \geq \overline{X}$. The system of differential equations is thus solved backwards from those terminal conditions. Cauchy-Lipschitz Theorem then applies and allows us to prove existence and uniqueness, of a solution with those boundary conditions at least locally while Wintner Theorem provides sufficient conditions that are satisfied by this system so as to ensure global existence. Intuitively, if the distribution has finite support, DM knows that, when the stock has passed $\overline{X}$, his action has no longer any impact on the arrival rate of a disaster. The myopic action is thus optimal from that point on. By backward induction, DM can reconstruct the unique action and payoff profile reaching that terminal point.

PROPOSITION 10 *Suppose that F has bounded support of the form $\left[0, \overline{X}\right]$ where $\overline{X}$ is finite. Then, there exists a unique continuously differentiable function, $\mathcal{V}^*(X)$, satisfying the functional equation (7.9) and the boundary condition*

$$(7.14) \quad \mathcal{V}^*(\overline{X}) = \mathcal{V}_\infty$$

.

Things are more complex when the distribution F has instead an infinite support because the above backward construction is no longer available. Yet, the Hartman-Grobman Theorem can be applied to the system of differential equations satisfied by $\mathcal{V}^*(X)$ and $\varphi(X)$. This Theorem shows that, in the neighborhood of the boundary conditions $\lim_{X\to+\infty} \mathcal{V}^*(X) = \lim_{X\to+\infty} \varphi(X) = \mathcal{V}_\infty$, there exists a unique stable manifold

[38]See Lemma D.4 in the Appendix.

in the $(\mathcal{V}, \varphi)$ space along which all solutions lie. From there, the local uniqueness of the time-consistent feedback rule $\sigma^*(X)$ and the *pseudo*-value function $\mathcal{V}^*(X)$ in the limit $X \to +\infty$ follows. In passing, this analysis provides interesting properties of the asymptotic behavior of these variables and allows us to derive rather tight bounds. In particular, the optimal action (which by definition cannot be conditioned on whether the tipping has been passed or not) now converges towards the myopic optimum while the value function still converges towards the corresponding myopic payoff.

PROPOSITION 11 *Suppose that F has unbounded support. Then, there exists a unique continuously differentiable function, $\mathcal{V}^*(X)$, satisfying the functional equation (7.9) and the boundary condition (7.10). $\mathcal{V}^*(X)$ and $\sigma^*(X)$ admit the following approximations when X is large:*

$$\mathcal{V}^*(X) - \mathcal{V}_\infty \approx_{+\infty} \frac{\Delta \mathcal{V}_\infty (1 - F(X))}{\zeta R'(0) + \lambda_0}, \tag{7.15}$$

$$\sigma^*(X) \approx_{+\infty} \zeta - \frac{\Delta \mathcal{V}_\infty f(X)}{\zeta R'(0) + \lambda_0}. \tag{7.16}$$

BOUNDS. As a by-product, our analysis of the existence of a *Stock-Markov Equilibrium* allows us to derive tight bounds on the *pseudo*-value function and the feed-back rule. These bounds are the same as in the scenario of Section 5. The dynamics is in fact quite similar. To illustrate, the upper bound on $\mathcal{V}^*(X)$ is readily obtained by following a non-equilibrium strategy consisting in adopting the myopic action under all circumstances.

PROPOSITION 12 *$\mathcal{V}^*(X)$ and $\sigma^*(X)$ admit the following bounds:*

$$\mathcal{V}_\infty < \mathcal{V}^*(X) < \mathcal{V}_\infty \left(1 + \frac{\Delta}{\lambda_0}(1 - F(X))\right) \quad \forall X \geq 0, \tag{7.17}$$

$$\sqrt{2\lambda_0 \mathcal{V}_\infty - 2D} < \sigma^*(X) < \zeta \quad \forall X \geq 0. \tag{7.18}$$

In the long run, the stock is likely to have gone through most possible values of the tipping point. The choice of the action then has almost no longer any influence on the arrival rate of a disaster which is almost surely θ_1. The optimal action thus necessarily converges towards the myopically optimal decision as shown in (7.16). At the same time, the *pseudo*-value function converges towards its value $\mathcal{V}_\infty$ under a myopic scenario.

8. THE VALUE OF THE *PRECAUTIONARY PRINCIPLE*

The *Precautionary Principle* imposes a legal restriction on the set of actions available to DM. As long as the accumulated stock remains low and few possible values of the tipping point might have already been passed, actions are capped. That actions remain below the cap can be legally enforced in a first phase. Later, DM will freely choose actions with no restriction beyond the equilibrium conditions embodied in a time-consistent plan. While there is the possibility to commit not to choose actions (at least for a while) beyond a fixed cap that is chosen *ex ante*, actions in the second phase depend on the existing stock and obey equilibrium requirements. The *Precautionary Principle* is then a device to restore some commitment, although a very imperfect one, since the cap is kept constant and does not evolve with time as the optimal trajectory would require.[39]

More precisely, suppose that DM is forced to choose a fixed action x over an interval of length $\varepsilon > 0$ where ε is not necessarily infinitesimal in contrast with our previous analysis of impulse deviations. The stock then reaches a level $x\varepsilon$ at the end of this first phase. Afterwards, DM is free to follow the equilibrium feedback rule $\sigma^*(X)$ for $X \geq x\varepsilon$. The benefit of such policy is to be able to commit to a given action over $[0, \varepsilon]$. The cost is that such a commitment is independent of where the stock lies during that interval while, even though it is imperfectly so, the time-consistent solution keeps track of such information.

Using the definition of an impulse deviation (7.6) and (7.7), we first write DM's intertemporal payoff $\mathcal{V}(x, \varepsilon; 0)$ in terms of (x, ε) for any arbitrary interval of length ε and any fixed action x chosen over $[0, \varepsilon]$, when starting from an initial stock of zero as

$$\mathcal{V}(x, \varepsilon; 0) = \int_0^{+\infty} e^{-\lambda_0 \tau} \mathcal{Z}(x, \varepsilon, \tau; 0) u(y(x, \varepsilon, \tau; 0)) d\tau. \tag{8.1}$$

From (D.5), the stock trajectory becomes

$$\widetilde{X}(x, \varepsilon, \tau; 0) = \begin{cases} x\tau & \text{if } \tau \in [0, \varepsilon], \\ X^*(\tau - \varepsilon; x\varepsilon) = x\varepsilon + \int_\varepsilon^\tau \sigma^*(X^*(s - \varepsilon; x\varepsilon)) ds & \text{if } \tau \geq \varepsilon. \end{cases} \tag{8.2}$$

We assume that the intertemporal payoff $\mathcal{V}(x, \varepsilon; 0)$ so obtained is a strictly quasi-concave function of (x, ε) so that first-order conditions for optimality are also sufficient. Of course, DM should optimize over the fixed action x and the length of the commitment phase ε. A first and intuitive result is that, at the optimum, the following smooth-pasting condition should hold:

$$x = \sigma^*(x\varepsilon). \tag{8.3}$$

A solution to this equation necessarily exists since $0 < \sqrt{2\lambda_0 \mathcal{V}_\infty - 2D} < \sigma^*(0) < \zeta$. From strict quasi-concavity, this solution denoted as $x^*(\varepsilon)$ is unique. Condition (8.3) just says that, at the end of the commitment period, DM should move continuously from his committed action to the equilibrium feedback rule that will be followed from that date on. If that equality were not to hold, it would have been optimal to extend or contract the length of the commitment period.

Our key finding is stated in the next proposition.

PROPOSITION 13 *A cap on actions for a fixed length of time $\varepsilon^* > 0$ is optimal if*

$$\dot{\sigma}^*(0) < 0.$$

We already know from Section 4.2 that the equilibrium action is decreasing when DM has complete information on the fact that the tipping point lies at a strictly positive

[39] Working with a Ramsey model of growth with a Chichilnisky welfare criterion to evaluate trajectories, Asheim and Ekeland (2015) shows that strategies in any equilibrium with non-commitment are such that there exists an upper bound on the destruction of natural capital. The limit on actions is thus derived from equilibrium behavior while it is imposed as a commitment device in our framework.

level of the stock. The same is true when DM believes that the tipping point is unlikely earlier on; i.e., when the distribution of possible tipping points puts little mass for low levels of that tipping point. To illustrate, observe that, taken together (7.9) and (7.12) and differentiating imply

$$\dot{\sigma}^*(0)\sigma^*(0) = \lambda_0\dot{\mathcal{V}}^*(0) + \Delta f(0)\varphi(0)$$

and the right-hand side is negative when $f(0)$ is small enough. Under those circumstances, actions are too high at the beginning of the trajectory if no constraints are imposed. The *Precautionary Principle* is a way of controlling DM's incentives to choose such actions.

9. CONCLUSION

This paper discusses the relevance of the *Precautionary Principle,* a controversial legal framework that aims at bounding decisions for certain periods of time. We argue that such a ban only makes sense when there exists a conflict of interests between selves of the decision-maker acting at different points in time. We propose a simple dynamic setting where a decision-maker controls actions whose cumulative stock over time increases the risk of an environmental catastrophe. Deep uncertainty on the location of the tipping point of the physical process generates a time-inconsistency problem. By generalizing Bellman techniques in a context where dynamic programming fails, we have characterized equilibrium paths of actions when the decision-maker can only commit for arbitrarily small lengths of time. This framework allows us to show under which conditions, imposing a *Precautionary Principle*, viewed as a commitment to a fixed action for a given period of time, helps.

Other models could potentially provide foundations to this *Principle*, especially when political considerations are at play. To illustrate, consider the possibility that rotating decision-makers with different preferences are democratically elected for periods of finite length. If a first decision-maker knows he is about to step down from power and be replaced with another decision-maker who cares less about the cost of a catastrophe (or has less power to decide, for example if the future involves a free trade agreement with countries that care less about the catastrophe) he might enact laws that stipulate to limit actions.[40] Now the *Precautionary Principle* is akin to a political constraint on future decision-makers. Although attractive, such political considerations would also suggest that a decision-maker who instead does not care much about the catastrophe should force more prudent followers to adopt a minimal level of actions. In fact, we do not observe such a *reverse Precautionary Principle*, which in our view also casts doubt on the validity of such political economy foundations for the *Precautionary Principle.*

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[40] Historical precedents include the Second Amendment to the U.S. Constitution (1791) which guarantees individual citizens' right to bear arms and has prevented reforms despite frequent political interventions; more recently Austria has written in its constitution its refusal of nuclear energy (1999). It will be interesting to follow potential trade disputes between the EU, where some rules are (loosely) based on the *Precautionary Principle*, and Canada, who refuses precautionary arguments, following the implementation of the CETA.

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APPENDIX A: KNOWN TIPPING POINT

PROOFS OF PROPOSITION 1 AND PROPOSITION 2: Take $X < X_0$ and fix ε small enough so that $X + x\varepsilon < X_0$. Denote $\mathcal{D}(\varepsilon) = \{x \text{ s.t. } X + x\varepsilon < X_0\}$. By the *Principle of Dynamic Programming* when applied to (4.4), we must have

$$\mathcal{V}^{ck}(X; X_0) \equiv \sup_{x \in \mathcal{D}(\varepsilon)} \int_0^{\varepsilon} e^{-\lambda_0 \tau} u(x) d\tau + e^{-\lambda_0 \varepsilon} \mathcal{V}^{ck}(X + x\varepsilon; X_0).$$

Taking first-order Taylor approximations when $\mathcal{V}^{ck}(X; X_0)$ is continuously differentiable in X, we may rewrite this problem as

$$\mathcal{V}^{ck}(X; X_0) = \sup_{x \in \mathcal{D}(\varepsilon)} \varepsilon u(x) + (1 - \lambda_0 \varepsilon)(\mathcal{V}^{ck}(X; X_0) + x\varepsilon \dot{\mathcal{V}}^*(X; X_0)).$$

The corresponding HBJ equation writes as

$$\text{(A.1)} \qquad \lambda_0 \mathcal{V}^{ck}(X; X_0) = \max_x x \dot{\mathcal{V}}^{ck}(X; X_0) - \frac{1}{2}(x - \zeta)^2 + \lambda_1 \mathcal{V}_\infty$$

together with the boundary condition (4.6).

The maximand of the right-hand side of (A.1) is obtained for the optimal feedback rule (4.7). Inserting this feedback rule into (A.1) yields

$$\text{(A.2)} \qquad \lambda_0 \mathcal{V}^{ck}(X; X_0) = \zeta \dot{\mathcal{V}}^{ck}(X; X_0) + \frac{(\dot{\mathcal{V}}^{ck}(X; X_0))^2}{2} + \lambda_1 \mathcal{V}_\infty.$$

Solving this second-degree polynomia for $\dot{\mathcal{V}}^{ck}(X; X_0)$ and taking the root ensuring that $\sigma^*(X)$ as given by (4.7) remains positive yields (4.5).

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$$\text{(A.3)} \qquad \widehat{\mathcal{V}}(X) = \frac{\lambda_1}{\lambda_0} \mathcal{V}_\infty.$$

From (4.5), we have $\dot{\mathcal{V}}^{ck}(X; X_0) \leq 0$ if and only if $\mathcal{V}^{ck}(X) \leq \widehat{\mathcal{V}}(X)$. Observe that $\mathcal{V}^{ck}(X_0; X_0) < \widehat{\mathcal{V}}(X_0)$ because of (4.6). Moreover, $\mathcal{V}^{ck}(X; X_0)$ were to cross $\widehat{\mathcal{V}}(X)$ at $X_1 < X_0$, we would have $\dot{\mathcal{V}}^{ck}(X_1; X_0) = 0$. Observe that $\widehat{\mathcal{V}}(X)$ is a constant solution to (4.5). Suppose that $\mathcal{V}^{ck}(X; X_0)$ were to cross $\widehat{\mathcal{V}}(X)$ at $X_1 < X_0$. By Cauchy-Lipschitz Theorem, the only solution to (4.5) which is such $\mathcal{V}^{ck}(X_1; X_0) = \widehat{\mathcal{V}}(X_1)$ is such that $\mathcal{V}^{ck}(X, X_0) = \widehat{\mathcal{V}}(X)$ for all $X \in [0, X_0]$. This would contradict the boundary condition (4.6). Hence, necessarily, $\mathcal{V}^{ck}(X; X_0)$ remains always below $\widehat{\mathcal{V}}(X)$ and the right-hand side inequality of (4.13) holds. From (4.5), it then follows that $\dot{\mathcal{V}}^{ck}(X; X_0) < 0$ for $X < X_0$. From (4.6), we thus have necessarily $\mathcal{V}^{ck}(X; X_0) > \mathcal{V}_\infty)$ for $X < X_0$ and the left-hand side inequality of (4.13) also holds.

Turning now to the optimal action. The right-hand side inequality of (4.14) follows from (4.7) and $\dot{\mathcal{V}}^{ck}(X; X_0) < 0$ for $X < X_0$. The left-hand side inequality follows from the left-hand side inequality in (4.13), together with (4.5) and (4.7).

Differentiating (A.2) with respect to X yields

$$(\text{A.4}) \qquad (\dot{\mathcal{V}}^{ck}(X; X_0) + \zeta)\ddot{\mathcal{V}}^{ck}(X; X_0) = \lambda_0 \dot{\mathcal{V}}^{ck}(X; X_0)$$

or

$$(\text{A.5}) \qquad \left(1 + \frac{\zeta}{\dot{\mathcal{V}}^{ck}(X; X_0)}\right) \ddot{\mathcal{V}}^{ck}(X; X_0) = \lambda_0.$$

Because $\dot{\mathcal{V}}^{ck}(X; X_0) < 0$ for $X \in [0, X_0)$ and $\sigma^{ck}(X; X_0) = \dot{\mathcal{V}}^{ck}(X; X_0) + \zeta > 0$, we deduce that $\ddot{\mathcal{V}}^{ck}(X; X_0) < 0$ for $X \in [0, X_0)$ and thus $\sigma^{ck}(X; X_0)$ is decreasing.

Verification Theorem. It is routine and thus omitted.

Q.E.D.

Next proposition presents some detailed analysis of the solution in the case where the tipping point is known.

Proposition A.1 *Suppose that the non-homogeneous Poisson process is defined by (3.1).*

- *The optimal action is decreasing over $t \in [0, t_0)$ with $x^{ck}(t; X_0) < x^m$ for all $t \in [0, t_0)$:*

$$(\text{A.6}) \qquad x^{ck}(t; X_0) = \begin{cases} \zeta - \left(\zeta - \sqrt{2\lambda_0 \mathcal{V}_\infty - 2D}\right) e^{\lambda_0 (t - t_0)} & \text{for } t \in [0, t_0), \\ \zeta & \text{for } t \geq t_0 \end{cases}$$

where t_0 (with $t_0 > t^m$), the date at which the tipping point is reached, is the unique positive root for

$$(\text{A.7}) \qquad \zeta t_0 - X_0 = \left(\zeta - \sqrt{2\lambda_0 \mathcal{V}_\infty - 2D}\right) \frac{1 - e^{-\lambda_0 t_0}}{\lambda_0}.$$

- *The optimal stock $X^{ck}(t; X_0)$ satisfies*

$$(\text{A.8}) \qquad X^{ck}(t; X_0) = \begin{cases} \zeta t - \left(\zeta - \sqrt{2\lambda_0 \mathcal{V}_\infty - 2D}\right) e^{-\lambda_0 t_0} \frac{e^{\lambda_0 t} - 1}{\lambda_0} & \text{for } t \in [0, t_0), \\ \zeta (t - t_0) + X_0 & \text{for } t \geq t_0. \end{cases}$$

Proof of Proposition A.1: Integrating (A.5) yields

$$(\text{A.9}) \qquad \dot{\mathcal{V}}^{ck}(X; X_0) + \zeta \log \left(\frac{\dot{\mathcal{V}}^{ck}(X; X_0)}{\dot{\mathcal{V}}^{ck}(0; X_0)} \right) = \lambda_0 X + \dot{\mathcal{V}}^{ck}(0; X_0).$$

The date t_0 at which a level of the stock X_0 is reached, starting from $X^{ck}(0; X_0) = 0$ along the trajectory

$$\dot{X}^{ck}(t; X_0) = \sigma^{ck}(X^{ck}(t; X_0), X_0) \tag{A.10}$$

is obtained as

$$t_0 = \int_0^{t_0} dt = \int_0^{X^{ck}(t_0;X_0)} \frac{d\widetilde{X}}{\sigma^{ck}(\widetilde{X}; X_0)} = \int_0^{X^{ck}(t_0;X_0)} \frac{d\widetilde{X}}{\zeta + \dot{\mathcal{V}}^{ck}(\widetilde{X}; X_0)}$$

where the last equality follows from (4.7). We rewrite this condition using (A.4) as

$$\lambda_0 t_0 = \int_0^{X^{ck}(t_0;X_0)} \frac{\ddot{\mathcal{V}}^{ck}(\widetilde{X}; X_0)}{\dot{\mathcal{V}}^{ck}(\widetilde{X}; X_0)} d\widetilde{X}$$

or

$$\lambda_0 t_0 = \log\left(\frac{\dot{\mathcal{V}}^{ck}(X^{ck}(t_0; X_0); X_0)}{\dot{\mathcal{V}}^{ck}(0; X_0)}\right)$$

and thus

$$\dot{\mathcal{V}}^{ck}(X^{ck}(t_0; X_0); X_0) = \dot{\mathcal{V}}^{ck}(0; X_0)e^{\lambda_0 t_0}. \tag{A.11}$$

From (A.10) and (A.11), we thus obtain

$$\dot{X}^{ck}(t_0; X_0) = \dot{\mathcal{V}}^{ck}(0; X_0)e^{\lambda_0 t_0} + \zeta. \tag{A.12}$$

Integrating yields

$$X^{ck}(t_0; X_0) = \frac{\dot{\mathcal{V}}^{ck}(0; X_0)}{\lambda_0}\left(e^{\lambda_0 t_0} - 1\right) + \zeta t_0. \tag{A.13}$$

Using (A.11), $X^{ck}(t_0; X_0) = X_0$ and $\mathcal{V}^{ck}(X_0; X_0) = \mathcal{V}_\infty$ also gives us

$$\dot{\mathcal{V}}^{ck}(0; X_0) = \dot{\mathcal{V}}^{ck}(X_0; X_0)e^{-\lambda_0 t_0} = \left(-\zeta + \sqrt{2\lambda_0\mathcal{V}_\infty - 2D}\right)e^{-\lambda_0 t_0} \tag{A.14}$$

Inserting this expression into (A.13) for $t = t_0$ yields

$$X_0 = \frac{-\zeta + \sqrt{2\lambda_0\mathcal{V}_\infty - 2D}}{\lambda_0}\left(1 - e^{-\lambda_0 t_0}\right) + \zeta t_0. \tag{A.15}$$

Therefore, the date t_0 at which the tipping point X_0 is reached is given by (A.7).

Actions and Monotonicity. Turning now to the optimal action profile, we have

$$x^{ck}(t; X_0) = \sigma^{ck}(X^{ck}(t; X_0); X_0) = \zeta + \dot{\mathcal{V}}^{ck}(X^{ck}(t; X_0); X_0),$$

Using (A.14), we obtain for $t < t_0$,

$$x^{ck}(t; X_0) = \zeta - \left(\zeta - \sqrt{2\lambda_0\mathcal{V}_\infty - 2D}\right)e^{\lambda_0(t-t_0)}. \tag{A.16}$$

Observe that $x^{ck}(t; X_0)$ is decreasing with t for $t \in (0, t_0)$, and constant thereafter.

Stock. We get for $t \leq t_0$

$$X^{ck}(t; X_0) = \zeta t - \left(\zeta - \sqrt{2\lambda_0\mathcal{V}_\infty - 2D}\right)e^{-\lambda_0 t_0}\frac{(e^{\lambda_0 t} - 1)}{\lambda_0}. \tag{A.17}$$

Moreover, for $t \geq t_0$, we have $x^{ck}(t; X_0) = \zeta$ and thus

$$X^{ck}(t; X_0) = X_0 + \zeta(t - t_0). \tag{A.18}$$

Gathering (A.17) and (A.18) yields (A.8).

Unicity of t_0. Consider

$$\delta(t) \equiv \left(\zeta - \sqrt{2\lambda_0 \mathcal{V}_\infty - 2D}\right) \frac{(1 - e^{-\lambda_0 t})}{\lambda_0} - \zeta t + X_0.$$

We have

$$\delta'(t) = \left(\zeta - \sqrt{2\lambda_0 \mathcal{V}_\infty - 2D}\right) e^{-\lambda_0 t} - \zeta \text{ and } \delta''(t) = -\lambda_0 \left(\zeta - \sqrt{2\lambda_0 \mathcal{V}_\infty - 2D}\right) e^{-\lambda_0 t} < 0.$$

Hence, δ is strictly concave and thus cross zero at most twice. Since, $\delta(0) = X_0 > 0$ and $\lim_{t_0 \to +\infty} \delta(t) = -\infty$, there is a unique positive root $t_0 \in \left(\frac{X_0}{\zeta}, +\infty\right)$ for (A.7). *Q.E.D.*

APPENDIX B: UNCERTAINTY ON THE TIPPING POINT

Proof of Lemma 1: The integral in the maximand on the right-hand side of (5.1) can be written as

$$\int_X^{+\infty} \left(\int_t^{\tilde{T}(\tilde{X};X,t)} e^{-r(\tau-t)} e^{-\theta_0(\tau-t)} u(x(\tau)) d\tau \right.$$

$$\left. + e^{-\lambda_0(\tilde{T}(\tilde{X};X,t)-t)} \int_{\tilde{T}(\tilde{X};X,t)}^{+\infty} e^{-r(\tau-\tilde{T}(\tilde{X};X,t))} e^{-\theta_1(\tau-\tilde{T}(\tilde{X};X,t))} \lambda_1 \mathcal{V}_\infty d\tau \right) f(\tilde{X}) d\tilde{X}$$

$$= \int_X^{+\infty} \left(\int_t^{\tilde{T}(\tilde{X};X,t)} e^{-\lambda_0(\tau-t)} u(x(\tau)) d\tau + e^{-\lambda_0(\tilde{T}(\tilde{X};X,t)-t)} \mathcal{V}_\infty \right) f(\tilde{X}) d\tilde{X}.$$

Integrating by parts this expression, we obtain:

$$(1 - F(X)) \int_t^{+\infty} e^{-\lambda_0(\tau-t)} u(x(\tau)) d\tau$$

$$- \int_X^{+\infty} (F(\tilde{X}) - F(X)) \frac{\partial \tilde{T}}{\partial X}(\tilde{X}; X, t) e^{-\lambda_0(\tilde{T}(\tilde{X};X,t)-t)} \left(u(x(\tilde{T}(\tilde{X}; X, t))) - \lambda_0 \mathcal{V}_\infty \right) d\tilde{X}.$$

Taking now time as the relevant variable to compute this last integral (i.e., setting $\tilde{X} = \tilde{X}(\tau; X, t) \Leftrightarrow \tau = \tilde{T}(\tilde{X}; X, t)$ with $d\tilde{X} = \dot{\tilde{X}}(\tau; X, t) d\tau = \frac{d\tau}{\frac{\partial \tilde{T}}{\partial X}(\tilde{X}(\tau;X,t);X,t)}$), we rewrite

$$\int_X^{+\infty} (F(\tilde{X}) - F(X)) \frac{\partial \tilde{T}}{\partial X}(\tilde{X}; X, t) e^{-\lambda_0(\tilde{T}(\tilde{X};X,t)-t)} \left(u(x(\tilde{T}(\tilde{X}; X, t))) - \lambda_0 \mathcal{V}_\infty \right) d\tilde{X} \tag{B.1}$$

$$= \int_X^{+\infty} (F(\tilde{X}) - F(X)) \frac{\partial \tilde{T}}{\partial X}(\tilde{X}; X, t) e^{-\lambda_0(\tilde{T}(\tilde{X};X,t)-t)} \left(u(\zeta) - \frac{1}{2}(x(\tau) - \zeta)^2 - \lambda_0 \mathcal{V}_\infty \right) d\tilde{X}$$

$$= \int_t^{+\infty} (F(\tilde{X}(\tau; X, t)) - F(X)) e^{-\lambda_0(\tau-t)} \left(\Delta \mathcal{V}_\infty - \frac{1}{2}(x(\tau) - \zeta)^2 \right) d\tau.$$

The maximand on the right-hand side of (5.1) is finally expressed as

$$(1-F(X))\int_t^{+\infty} e^{-\lambda_0(\tau-t)}\left(\lambda_1\mathcal{V}_\infty-\frac{1}{2}(x(\tau)-\zeta)^2\right)d\tau$$

$$-\int_t^{+\infty}(F(\tilde{X}(\tau;X,t))-F(X))e^{-\lambda_0(\tau-t)}\left(\Delta\mathcal{V}_\infty-\frac{1}{2}(x(\tau)-\zeta)^2\right)d\tau,$$

or

$$(1-F(X))\mathcal{V}_\infty+\int_t^{+\infty}(1-F(\tilde{X}(\tau;X,t)))e^{-\lambda_0(\tau-t)}\left(\Delta\mathcal{V}_\infty-\frac{1}{2}(x(\tau)-\zeta)^2\right)d\tau.$$

Changing variables in the integrand yields

$$\text{(B.2)}\qquad (1-F(X))\mathcal{V}_\infty+\int_0^{+\infty}(1-F(\tilde{X}(\tau+t;X,0)))e^{-\lambda_0\tau}\left(\Delta\mathcal{V}_\infty-\frac{1}{2}(x(\tau+t)-\zeta)^2\right)d\tau.$$

Now, we notice that, if the trajectory $\tilde{X}(\tau;X,t)$ with the associated actions $\tilde{x}(\tau;X,t)=\frac{\partial\tilde{X}}{\partial\tau}(\tau;X,t)$ for $\tau\geq t$ were to maximize the right-hand side of (B.3), the trajectory $\tilde{X}(\tau'+t;X,0)$ with the associated actions $\tilde{x}(\tau'+t;X,0)=\frac{\partial\tilde{X}}{\partial\tau'}(\tau'+t;X,0)$ would achieve the maximand for (B.2). Hence, we can rewrite the maximization problem as

$$\text{(B.3)}\qquad (1-F(X))\tilde{\mathcal{V}}_2^u(X,t)=\sup_{\mathbf{x}_t,\tilde{X}(\cdot)\text{ s.t. (4.2)}}(1-F(X))\mathcal{V}_\infty$$

$$+\int_0^{+\infty}e^{-\lambda_0\tau}(1-F(\tilde{X}(\tau;X))\left(\Delta\mathcal{V}_\infty-\frac{1}{2}(x(\tau)-\zeta)^2\right)d\tau\quad\forall(X,t).$$

It immediately follows from (B.3) that we can look for a solution of the form $\tilde{\mathcal{V}}_2^u(X,t)=\mathcal{V}^u(X)$ where $\mathcal{V}^u(X)$ is defined as

(B.4)

$$(1-F(X))(\mathcal{V}^u(X)-\mathcal{V}_\infty)\equiv\sup_{\mathbf{x},\widetilde{X}(\cdot)\text{ s.t. (4.3)}}\int_0^{+\infty}e^{-\lambda_0\tau}\left(1-F\left(\widetilde{X}(\tau;X)\right)\right)\left(\Delta\mathcal{V}_\infty-\frac{1}{2}(x(\tau)-\zeta)^2\right)d\tau.$$

After manipulations, we find

(B.5)

$$(1-F(X))\mathcal{V}^u(X)\equiv\sup_{\mathbf{x},\widetilde{X}(\cdot)\text{ s.t. (4.3)}}\left(1-F(X)-\lambda_0\int_0^{+\infty}e^{-\lambda_0\tau}\left(1-F\left(\widetilde{X}(\tau;X)\right)d\tau\right)\right)\mathcal{V}_\infty$$

$$+\int_0^{+\infty}e^{-\lambda_0\tau}\left(1-F\left(\widetilde{X}(\tau;X)\right)\right)u(x(\tau))d\tau.$$

Integrating by parts

$$1-F(X)-\lambda_0\int_0^{+\infty}e^{-\lambda_0\tau}\left(1-F\left(\widetilde{X}(\tau;X)\right)\right)d\tau=\int_0^{+\infty}e^{-\lambda_0\tau}f\left(\widetilde{X}(\tau;X)\right)x(\tau)d\tau.$$

Inserting into (B.5) yields (5.2). *Q.E.D.*

PROOF OF PROPOSITION 3: Define $\mathcal{W}^u(X) = (1 - F(X))(\mathcal{V}^u(X) - \mathcal{V}_\infty)$. It solves

$$\text{(B.6)} \qquad \mathcal{W}^u(X) \equiv \sup_{\mathbf{x}, \widetilde{X}(\cdot) \text{ s.t. } (4.3)} \int_0^{+\infty} e^{-\lambda_0 \tau} \left(1 - F\left(\widetilde{X}(\tau; X)\right)\right) \left(\Delta \mathcal{V}_\infty - \frac{1}{2}(x(\tau) - \zeta)^2\right) d\tau.$$

Existence of $\mathcal{W}^u(X)$ (not necessarily continuously differentiable) easily follows from Ekeland and Turnbull (1983, Corollary 2, p.92).[41] When continuously differentiable, $\mathcal{W}^u(X)$ satisfies the following HBJ equation:

$$\text{(B.7)} \qquad \lambda_0 \mathcal{W}^u(X) = \sup_{x \in \mathcal{X}} \left\{ x \dot{\mathcal{W}}^u(X) + (1 - F(X)) \left(\Delta \mathcal{V}_\infty - \frac{1}{2}(x - \zeta)^2\right) \right\}$$

and simplifying as

$$\text{(B.8)} \qquad \lambda_0 \mathcal{W}^u(X) = \zeta \dot{\mathcal{W}}^u(X) + \frac{(\dot{\mathcal{W}}^u(X))^2}{2(1 - F(X))} + \Delta \mathcal{V}_\infty (1 - F(X)).$$

Taking the highest root of this second-degree equation in $\dot{\mathcal{W}}^u(X)$ (so as to ensure that the feed-back rule defined in (B.10) below remains positive leading to a stock profile which is increasing over time), we rewrite this ordinary differential equation as

$$\text{(B.9)} \qquad \dot{\mathcal{W}}^u(X) = (1 - F(X)) \left(-\zeta + \sqrt{2\lambda_0 \mathcal{V}_\infty - 2D + 2\lambda_0 \frac{\mathcal{W}^u(X)}{1 - F(X)}} \right)$$

that can finally be written as (5.3).

FEEDBACK RULE. Maximizing the right-hand side of (B.7) yields

$$\text{(B.10)} \qquad \sigma^u(X) = \zeta + \frac{\dot{\mathcal{W}}^u(X)}{1 - F(X)}$$

which can finally be written as (5.5).

BOUNDARY CONDITION. From (B.4) and the fact that $1 - F(X + \int_0^t x(\tau) d\tau)$ converges towards zero for any non-negative action profile when $X \to +\infty$ we obtain

$$\text{(B.11)} \qquad \lim_{X \to +\infty} \mathcal{W}^u(X) = 0.$$

Q.E.D.

PROOF OF PROPOSITION 4: Consider (B.9). Let denote the locus of points where $\dot{\mathcal{W}}^u(X) = 0$ as

$$\widehat{\mathcal{W}}(X) = (1 - F(X)) \frac{\Delta}{\lambda_0} \mathcal{V}_\infty.$$

Observe that, $\widehat{\mathcal{W}}(0) = \frac{\Delta}{\lambda_0} \mathcal{V}_\infty > 0$, $\widehat{\mathcal{W}}(X)$ is decreasing and goes to 0 as X goes to $+\infty$.

EXISTENCE. Consider the domain $\mathcal{D} = \{(X, W) | \widehat{\mathcal{W}}(X) \geq W \geq 0 \text{ for some } X \geq 0\}$. The boundaries of $\mathcal{D}$ is made of the vertical segment $W \in [0, \widehat{\mathcal{W}}(0)]$, the horizontal axis $\{W = 0\}$ and the curve $\{W = \widehat{\mathcal{W}}(X), X \geq 0\}$. Let the flow defined by (B.9) be $\gamma : (\mathcal{W}_0, X) \to \mathcal{W}(X|\mathcal{W}_0)$ where $\mathcal{W}(X|\mathcal{W}_0)$ is the solution to (5.3) for some fixed initial value $\mathcal{W}_0$. This flow is of course

[41] Similar existence arguments can be used throughout the paper and won't be repeated.

continuous. By construction, any solution $\mathcal{W}(X|\mathcal{W}_0)$ that crosses $\widehat{\mathcal{W}}(X)$ at some $X_1 \geq 0$ is such that $\mathcal{W}(X|\mathcal{W}_0)$ is decreasing for $X < X_1$ and increasing for $X > X_1$ and thus can only cross $\widehat{\mathcal{W}}(X)$ once. Hence, such solution cannot satisfy the boundary condition (B.11). Take any solution $\mathcal{W}(X|\mathcal{W}_0)$ that crosses the horizontal axis $\{\mathcal{W} = 0\}$ for some $X_2 \geq 0$. At such point, (B.9) indicates that $\dot{\mathcal{W}}(X_2|\mathcal{W}_0) < 0$. Such solution cannot converge towards 0 either since, otherwise, there would exist a point $X_3 > X_2$ such that $\dot{\mathcal{W}}(X_3|\mathcal{W}_0) = 0$ and $\mathcal{W}(X_3|\mathcal{W}_0) < 0$. At such point, we should also have $\mathcal{W}(X_3|\mathcal{W}_0) = \widehat{\mathcal{W}}(X_3)$ which yields a contradiction with $\mathcal{W}(X_3|\mathcal{W}_0) < 0 < \widehat{\mathcal{W}}(X_3)$.

From these observations, and from the continuity of the flow γ, we deduce that the reciprocal image of the horizontal line $\{W = 0\}$ is of the form $[0, W_{02})$. Similarly, the reciprocal image of $\{W = \widehat{\mathcal{W}}(X), X \geq 0\}$ is of the form $(W_{01}, \widehat{\mathcal{W}}(0)]$ with necessarily $W_{02} \leq W_{01}$. Of course, $[0, W_{02})$ and $(W_{01}, \widehat{\mathcal{W}}(0)]$ cannot overlap because it would violate the local uniqueness of the solution $\mathcal{W}(X|\mathcal{W}_0)$ to (5.3) around $X = 0$ (Cauchy-Lipschitz Unicity Theorem). Thus $[W_{02}, W_{01}]$ is non-empty and necessarily a solution with $\mathcal{W}_0 \in [W_{02}, W_{01}]$ is such that:

$$\lim_{X\to+\infty} \mathcal{W}(X|\mathcal{W}_0) = 0.$$

This proves existence of a solution $\mathcal{W}^u(X)$ to (B.9) that satisfies the boundary condition (B.11).

Uniqueness. To prove uniqueness of the solution to (5.3) with the boundary condition (5.4), consider two putative distinct solutions to (5.3), say $\mathcal{W}_1$ and $\mathcal{W}_2$ satisfying this boundary condition with $\mathcal{W}_1(0) \in [W_{02}, W_{01}]$ and $\mathcal{W}_2(0) \in [W_{02}, W_{01}]$. Denote $\Delta\mathcal{W} = \mathcal{W}_1 - \mathcal{W}_2$ and suppose w.l.o.g that $\Delta\mathcal{W}(0) > 0$. Observe that necessarily $\Delta\mathcal{W}(X) > 0$ for all $X \geq 0$ (otherwise there would be a contradiction with Cauchy-Lipschitz Unicity Theorem at a putative date X_4 where $\mathcal{W}_1(X_4) = \mathcal{W}_2(X_4)$ would be supposed). We may compute:

$$\Delta\dot{\mathcal{W}}(X) = \frac{2\lambda_0 \Delta\mathcal{W}(X)}{\sqrt{2\lambda_0\mathcal{V}_\infty - 2D + 2\lambda_0\frac{\mathcal{W}_1(X)}{1-F(X)}} + \sqrt{2\lambda_0\mathcal{V}_\infty - 2D + 2\lambda_0\frac{\mathcal{W}_2(X)}{1-F(X)}}}.$$

Integrating, we get:

$$\Delta\mathcal{W}(X) = \Delta\mathcal{W}(0)e^{\int_0^X \frac{2\lambda_0 d\widetilde{X}}{\sqrt{2\lambda_0\mathcal{V}_\infty - 2D + 2\lambda_0\frac{\mathcal{W}_1(\widetilde{X})}{1-F(\widetilde{X})}} + \sqrt{2\lambda_0\mathcal{V}_\infty - 2D + 2\lambda_0\frac{\mathcal{W}_2(\widetilde{X})}{1-F(\widetilde{X})}}}}.$$

Observe that both $\mathcal{W}_1$ and $\mathcal{W}_2$ satisfy

$$0 < \mathcal{W}_i(X) < \widehat{\mathcal{W}}(X)$$

for $i = 1, 2$ when $\mathcal{W}_1(0) \in [W_{02}, W_{01}]$ and $\mathcal{W}_2(0) \in [W_{02}, W_{01}]$. It implies that

$$\int_0^X \frac{2\lambda_0 d\widetilde{X}}{\sqrt{2\lambda_0\mathcal{V}_\infty - 2D + 2\lambda_0\frac{\mathcal{W}_1(\widetilde{X})}{1-F(\widetilde{X})}} + \sqrt{2\lambda_0\mathcal{V}_\infty - 2D + 2\lambda_0\frac{\mathcal{W}_2(\widetilde{X})}{1-F(\widetilde{X})}}} \geq \frac{\lambda_0}{\zeta}X.$$

Hence, $\Delta\mathcal{W}(0) > 0$ also implies

$$\lim_{X\to+\infty} \Delta\mathcal{W}(X) = +\infty.$$

A contradiction with our assumption that $\mathcal{W}_1$ and $\mathcal{W}_2$ both satisfy the boundary condition (5.4). It follows that there exists a unique solution to (5.3) satisfying (5.4).

COMPARATIVE STATICS. The above analysis shows that any solution $\mathcal{W}^u(X)$ also satisfies

$$0 < \mathcal{W}^u(X) < \widehat{\mathcal{W}}(X). \tag{B.12}$$

From this, it immediately follows that (5.6).

Inserting now (B.12) into (B.9), we obtain

$$(1-F(X))\left(-\zeta+\sqrt{2\lambda_0\mathcal{V}_\infty-2D}\right) < \dot{\mathcal{W}}^u(X) < 0.$$

Inserting into (B.10) yields (5.7).

A VERIFICATION THEOREM. Proposition B.1 shows that the conditions given Proposition 4 to characterize a value function by means of an HBJ equation together with a boundary conditions are in fact sufficient. We follow Ekeland and Turnbull (1983, Theorem 1, p. 6) and derive a *Verification Theorem.*

PROPOSITION B.1 *Assume first that there exists a continuously differentiable function $\mathcal{W}_0(X)$ which satisfies:*

$$\lambda_0\mathcal{W}_0(X) \geq x\dot{\mathcal{V}}_0(X) + (1-F(X))\left(\Delta\mathcal{V}_\infty - \frac{1}{2}(x-\zeta)^2\right) \quad \forall (x,X); \tag{B.13}$$

and, second, that there exists an action profile $\mathbf{x}_0$ and $X_0(t)=\int_0^t x_0(\tau)d\tau$ such that

$$\lambda_0\mathcal{W}_0(X_0(t)) = x_0(t)\dot{\mathcal{V}}_0(X_0(t)) + (1-F(X_0(t)))\left(\Delta\mathcal{V}_\infty - \frac{1}{2}(x_0(t)-\zeta)^2\right) \quad \forall t\geq 0. \tag{B.14}$$

Then $\mathbf{x}_0$ is an optimal action profile with its associated path $X_0(t)$.

PROOF OF PROPOSITION B.1: First observe that $\mathcal{W}^u(X)$ as characterized in the Proof of Proposition 4 is continuously differentiable. It is our candidate for the above function $\mathcal{W}_0(X)$. By definition (B.7), we have

$$\lambda_0\mathcal{W}^u(X) \geq x\dot{\mathcal{W}}^u(X) + (1-F(X))\left(\Delta\mathcal{V}_\infty - \frac{1}{2}(x-\zeta)^2\right) \quad \forall (x,X). \tag{B.15}$$

To get (B.14), we use again (B.7) but now applied to the path $(x_0(t),X_0(t)) \equiv (x^u(t),X^u(t))$ where $X^u(t)=\int_0^t \sigma^u(X^u(\tau))d\tau$ and $x^u(t)=\sigma^u(X^u(t))$.

Define now a value function as $\widetilde{\mathcal{W}}^u(X,t)=e^{-\lambda_0 t}\mathcal{W}^u(X)$. By (B.15), we get

$$0 \geq \frac{\partial\widetilde{\mathcal{W}}^u}{\partial t}(X,t) + x\frac{\partial\widetilde{\mathcal{W}}^u}{\partial X}(X,t) + e^{-\lambda_0 t}(1-F(X))\left(\Delta\mathcal{V}_\infty - \frac{1}{2}(x-\zeta)^2\right) \quad \forall (x,X). \tag{B.16}$$

Using $x^u(t)=\sigma^u(X^u(t))$ and (B.14), we also get

(B.17)
$$0 = \frac{\partial\widetilde{\mathcal{W}}^u}{\partial t}(X^u(t),t) + x^u(t)\frac{\partial\widetilde{\mathcal{W}}^u}{\partial X}(X^u(t),t) + e^{-\lambda_0 t}(1-F(X^u(t)))\left(\Delta\mathcal{V}_\infty - \frac{1}{2}(x^u(t)-\zeta)^2\right) \quad \forall t\geq 0.$$

Take now an arbitrary action plan $\mathbf{x}$ with the associated path $X(t)=\int_0^t x(\tau)d\tau$. Let us fix an arbitrary $T>0$. Integrating (B.16) along the path $(x(t),X(t))$, we compute

$$0 \geq \int_0^T \left(\frac{\partial\widetilde{\mathcal{W}}^u}{\partial t}(X(t),t) + x(t)\frac{\partial\widetilde{\mathcal{W}}^u}{\partial X}(X(t),t) + e^{-\lambda_0 t}(1-F(X(t)))\left(\Delta\mathcal{V}_\infty - \frac{1}{2}(x(t)-\zeta)^2\right)\right)dt$$

or

$$0 \geq \int_0^T \left(\frac{d\widetilde{\mathcal{W}^u}}{dt}(X(t),t) + e^{-\lambda_0 t}(1 - F(X(t))) \left(\Delta\mathcal{V}_\infty - \frac{1}{2}(x(t)-\zeta)^2 \right) \right) dt \quad \forall T \geq 0$$

By definition of the total derivative $\frac{d\widetilde{\mathcal{W}^u}}{dt}(X(t),t)$, we thus get

$$\widetilde{\mathcal{W}}^u(0,0) \geq \widetilde{\mathcal{W}}^u(X(T),T) + \int_0^T e^{-\lambda_0 t}(1 - F(X(t))) \left(\Delta\mathcal{V}_\infty - \frac{1}{2}(x(t)-\zeta)^2 \right) dt \quad \forall T \geq 0.$$

Because $\widetilde{\mathcal{W}}^u(X(T),T) = e^{-\lambda_0 T}\mathcal{W}^u(X(T))$ for all T, we obtain:

$$\widetilde{\mathcal{W}}^u(0,0) \geq e^{-\lambda_0 T}\mathcal{W}^u(X(T)) + \int_0^T e^{-\lambda_0 t}(1 - F(X(t))) \left(\Delta\mathcal{V}_\infty - \frac{1}{2}(x(t)-\zeta)^2 \right) dt \quad \forall T \geq 0$$

and, because $\mathcal{W}^u(X(T)) \geq 0$ for all T from our previous findings, we get

$$\widetilde{\mathcal{W}}^u(0,0) \geq \int_0^T e^{-\lambda_0 t}(1 - F(X(t))) \left(\Delta\mathcal{V}_\infty - \frac{1}{2}(x(t)-\zeta)^2 \right) dt \quad \forall T \geq 0.$$

Because $|e^{-\lambda_0 t}(1 - F(X(t))) \left(\Delta\mathcal{V}_\infty - \frac{1}{2}(x(t)-\zeta)^2 \right)| \leq Me^{-\lambda_0 t}$ for some M when $x \in \mathcal{X}$ the above integral converges for any feasible path $(x(t), X(t))$ as T goes to $+\infty$. Hence, we can write

$$\widetilde{\mathcal{W}}^u(0,0) \geq \int_0^{+\infty} e^{-\lambda_0 t}(1 - F(X(t))) \left(\Delta\mathcal{V}_\infty - \frac{1}{2}(x(t)-\zeta)^2 \right) dt.$$

Moreover, integrating (B.17), the inequality above is indeed an equality for $(x^u(t), X^u(t))$:

$$\widetilde{\mathcal{W}}^u(0,0) = \int_0^{+\infty} e^{-\lambda_0 t}(1 - F(X^u(t))) \left(\Delta\mathcal{V}_\infty - \frac{1}{2}(x^u(t)-\zeta)^2 \right) dt.$$

Thus $(x^u(t), X^u(t))$ is indeed an optimal path. *Q.E.D.*

Q.E.D.

Proof of Proposition 5: The *HBJ* equation (B.8) now writes as:

$$\lambda_0 \mathcal{W}^u(X)e^{-kX} = \frac{1}{2}(\dot{\mathcal{W}}^u(X))^2 + \zeta e^{-kX}\dot{\mathcal{W}}^u(X) + \Delta\mathcal{V}_\infty e^{-2kX}. \tag{B.18}$$

This expression suggests looking for a solution of the form

$$\mathcal{W}^u(X) = \alpha^* e^{-kX}$$

for some $\alpha^* > 0$. Inserting into (B.18), it is immediate to check that such α^* is a root to the following second-order equation:

$$\frac{\alpha^{*2}}{2} - \left(\frac{\lambda_0}{k^2} + \frac{\zeta}{k} \right) \alpha^* + \frac{\Delta\mathcal{V}_\infty}{k^2} = 0.$$

To ensure that the stock $X(t)$ is an increasing function, we select the lowest non-negative root, namely

$$\alpha^* = \frac{\lambda_0}{k^2} + \frac{\zeta}{k} - \sqrt{\left(\frac{\lambda_0}{k^2} + \frac{\zeta}{k} \right)^2 - 2\frac{\Delta\mathcal{V}_\infty}{k^2}}. \tag{B.19}$$

From there, (5.8), (5.9) and (5.10) immediately follow. *Q.E.D.*

PROOF OF PROPOSITION 6: To look at the long-run behavior, we first change variables and define

$$Y = 1 - F(X) \in [0,1]\,, \mathcal{W}^u(X) = \omega(Y), R(Y) = f(F^{-1}(1-Y))$$

From this, we get

$$\dot{\mathcal{W}}^u(X) = -\dot{\omega}(Y)R(Y).$$

Inserting into (B.9) yields

$$\text{(B.20)}\quad \dot{\omega}(Y) = \frac{Y}{R(Y)}\left(\zeta - \sqrt{2\lambda_0\mathcal{V}_\infty - 2D + 2\lambda_0\frac{\omega(Y)}{Y}}\right).$$

From (B.11), we deduce that

$$\text{(B.21)}\quad \omega(0) = 0.$$

Observe that $R(0) = 0$ and $\dot{R}(0) = \lim_{Y\to 0}\frac{R(Y)}{Y}$. Taking the limit of (B.20) to $(Y = 0, \omega(0) = 0)$, we find that $\dot{\omega}(0)$ must solve

$$\dot{\omega}(0) = \frac{1}{\dot{R}(0)}\left(\zeta - \sqrt{2\lambda_0\mathcal{V}_\infty - 2D + 2\lambda_0\dot{\omega}(0)}\right).$$

After manipulations, we find that $\dot{\omega}(0)$ must solve (B.19) for $k = \dot{R}(0)$. From this, (5.12) immediately follows. The right-hand side inequality in (5.12) follows from

$$\sqrt{\left(\frac{\lambda_0}{\dot{R}(0)} + \zeta\right)^2 - 2\Delta\mathcal{V}_\infty} - \frac{\lambda_0}{\dot{R}(0)} < \sqrt{\left(\frac{\lambda_0}{\dot{R}(0)} + \zeta\right)^2} - \frac{\lambda_0}{\dot{R}(0)} < \zeta.$$

Further, notice that

$$\frac{\mathcal{W}^u(X)}{1 - F(X)} = \frac{\omega(Y)}{Y}$$

so

$$\lim_{X\to\infty}\frac{\mathcal{W}^u(X)}{1 - F(X)} = \dot{\omega}(0)$$

which yields (5.11). *Q.E.D.*

APPENDIX C: DEEP UNCERTAINTY, TIME INCONSISTENCY

Main Results

PROOF OF LEMMA 2: Integrating by parts, we obtain:

$$\int_0^{\widetilde{X}(t;X)} \theta_1 e^{-(\theta_0\widetilde{T}(\widetilde{X};X)+\theta_1(t-\widetilde{T}(\widetilde{X};X))} f(\widetilde{X})d\widetilde{X} =$$

$$F(\widetilde{X}(t;X))\theta_1 e^{-\theta_0 t} - \Delta\int_0^{\widetilde{X}(t;X)} F(\widetilde{X})\frac{\partial\widetilde{T}}{\partial\widetilde{X}}(\widetilde{X};X)\theta_1 e^{-(\theta_0\widetilde{T}(\widetilde{X};X)+\theta_1(t-\widetilde{T}(\widetilde{X};X)))}d\widetilde{X}.$$

Now changing variables and setting $\widetilde{X} = \widetilde{X}(\tau; X)$ (with $d\widetilde{X} = \frac{\partial X}{\partial \tau}(\tau; X)d\tau$) in the integral, we obtain:

$$F(\widetilde{X}(t;X))\theta_1 e^{-\theta_0 t} - \Delta \int_0^{\widetilde{X}(t;X)} F(\widetilde{X}) \frac{\partial \widetilde{T}}{\partial \widetilde{X}}(\widetilde{X};X)\theta_1 e^{-(\theta_0 \widetilde{T}(\widetilde{X};X)+\theta_1(t-\widetilde{T}(\widetilde{X};X)))} d\widetilde{X}$$

$$= \theta_1 e^{-\theta_0 t}\left(F(\widetilde{X}(t;X)) - \Delta e^{-\Delta t}\int_0^t F(\widetilde{X}(\tau;X))e^{\Delta\tau}d\tau\right).$$

From this, it follows that:

$$g(t;\bar{\mathbf{x}}^t, X) = (1-F(\widetilde{X}(t;X))\theta_0 e^{-\theta_0 t} + \theta_1 e^{-\theta_0 t}\left(F(\widetilde{X}(t;X)) - \Delta e^{-\Delta t}\int_0^t F(\widetilde{X}(\tau;X))e^{\Delta\tau}d\tau\right).$$

Integrating by parts, we finally obtain:

$$g(t;\bar{\mathbf{x}}^t, X) = (1 - F(\widetilde{X}(t;X)))\theta_0 e^{-\theta_0 t} + \theta_1 e^{-\theta_1 t}\int_0^t f(\widetilde{X}(\tau;X))\frac{\partial \widetilde{X}}{\partial \tau}(\tau;X)e^{\Delta\tau}d\tau.$$

Integrating, we obtain:

$$G(t;\bar{\mathbf{x}}^t, X) = \int_0^t \left((1 - F(\widetilde{X}(\tau;X))\theta_0 e^{-\theta_0 \tau} + \theta_1 e^{-\theta_1 \tau}\int_0^\tau f(\widetilde{X}(s;X))\frac{\partial \widetilde{X}}{\partial s}(s;X)e^{\Delta s}ds\right)d\tau$$

Integrating by parts and simplifying yields

$$1 - G(t;\bar{\mathbf{x}}^t, X) = (1 - F(\widetilde{X}(t;X))e^{-\theta_0 t} + e^{-\theta_1 t}\int_0^t f(\widetilde{X}(\tau;X))\frac{\partial X}{\partial \tau}(\tau;X)e^{\Delta\tau}d\tau.$$

Integrating by parts the last term, we now obtain

$$1 - G(t;\bar{\mathbf{x}}^t, X) = (1-F(\widetilde{X}(t;X))e^{-\theta_0 t} + e^{-\theta_1 t}\left(F(\widetilde{X}(t;X))e^{\Delta t} - \Delta\int_0^t F(\widetilde{X}(\tau;X))e^{\Delta\tau}d\tau\right)$$

which finally simplifies as (6.1). *Q.E.D.*

PROOF OF LEMMA 3: Observe that the first integral in the maximand on the right-hand side of (6.3) is

$$\mathcal{I}_1 = \int_0^X \left(\int_t^{+\infty} e^{-r(\tau-t)}e^{-\theta_1(\tau-t)}u(x(\tau))d\tau\right) f(\widetilde{X})d\widetilde{X} = F(X)\int_t^{+\infty} e^{-\lambda_1(\tau-t)}u(x(\tau))d\tau.$$

The second integral on right-hand side of (6.3) is

$$\mathcal{I}_2 = \int_X^{+\infty}\left(\int_t^{\widetilde{T}(\widetilde{X};X,t)} e^{-r(\tau-t)}e^{-\theta_0(\tau-t)}u(x(\tau))d\tau\right.$$

$$\left. + e^{-\theta_0(\widetilde{T}(\widetilde{X};X,t)-t)}\int_{\widetilde{T}(\widetilde{X};X,t)}^{+\infty} e^{-r(\tau-t)}e^{-\theta_1(\tau-\widetilde{T}(\widetilde{X};X,t))}u(x(\tau))d\tau\right)f(\widetilde{X})d\widetilde{X}.$$

Integrating by parts, we obtain:

$$\mathcal{I}_2 = (1 - F(X))\int_t^{\infty} e^{-r(\tau-t)}e^{-\theta_0(\tau-t)}u(x(\tau))d\tau$$

$$-\int_X^{+\infty}(F(\widetilde{X})-F(X))\frac{\partial\widetilde{T}}{\partial\widetilde{X}}(\widetilde{X};X,t)\Bigg(e^{-r(\widetilde{T}(\widetilde{X};X,t)-t)}e^{-\theta_0(\widetilde{T}(\widetilde{X};X,t)-t)}u(x(\widetilde{T}(\widetilde{X};X,t)))$$
$$-e^{-r(\widetilde{T}(\widetilde{X};X,t)-t)}e^{-\theta_0(\widetilde{T}(\widetilde{X};X,t)-t)}u(x(\widetilde{T}(\widetilde{X};X,t)))$$
$$+\Delta e^{-\theta_0(\widetilde{T}(\widetilde{X};X,t)-t)}\int_{\widetilde{T}(\widetilde{X};X,t)}^{+\infty}e^{-r(\tau-t)}e^{-\theta_1(\tau-\widetilde{T}(\widetilde{X};X,t))}u(x(\tau))d\tau\Bigg)d\widetilde{X}.$$

Simplifying, we get

$$\mathcal{I}_2=(1-F(X))\int_t^{\infty}e^{-\lambda_0(\tau-t)}u(x(\tau))d\tau$$
$$-\Delta\int_X^{+\infty}(F(\widetilde{X})-F(X))\frac{\partial\widetilde{T}}{\partial\widetilde{X}}(\widetilde{X};X,t)e^{-\theta_0(\widetilde{T}(\widetilde{X};X,t)-t)}\left(\int_{\widetilde{T}(\widetilde{X};X,t)}^{+\infty}e^{-r(\tau-t)}e^{-\theta_1(\tau-\widetilde{T}(\widetilde{X};X,t))}u(x(\tau))d\tau\right)d\widetilde{X}.$$

Taking now time as the relevant variable to compute the last integral in $\mathcal{I}_2$, i.e., setting

$$\widetilde{X}=\widetilde{X}(\tau;X,t)\Leftrightarrow\tau=\widetilde{T}(\widetilde{X};X,t)$$

with

$$d\widetilde{X}=\dot{\widetilde{X}}(\tau;X,t)d\tau=\frac{d\tau}{\frac{\partial\widetilde{T}}{\partial\widetilde{X}}(\widetilde{X}(\tau;X,t);X,t)}.$$

We thus rewrite

(C.1)
$$\int_X^{+\infty}(F(\widetilde{X})-F(X))\frac{\partial\widetilde{T}}{\partial\widetilde{X}}(\widetilde{X};X,t)e^{-\theta_0(\widetilde{T}(\widetilde{X};X,t)-t)}\left(\int_{\widetilde{T}(\widetilde{X};X,t)}^{+\infty}e^{-r(s-t)}e^{-\theta_1(s-\widetilde{T}(\widetilde{X};X,t))}u(x(s))ds\right)d\widetilde{X}$$
$$=\int_t^{+\infty}(F(\widetilde{X}(\tau;X,t))-F(X))e^{-\theta_0(\tau-t)}\left(\int_\tau^{+\infty}e^{-r(s-t)}e^{-\theta_1(s-\tau)}u(x(s))ds\right)d\tau$$
$$=\int_t^{+\infty}(F(\widetilde{X}(\tau;X,t))-F(X))e^{-\lambda_0(\tau-t)}\left(\int_\tau^{+\infty}e^{-\lambda_1(s-\tau)}u(x(s))ds\right)d\tau.$$

Summing $\mathcal{I}_1$ and $\mathcal{I}_2$ gives an expression of the maximand on the right-hand side of (6.3) as

(C.2)
$$\int_t^{+\infty}\Bigg(\left(F(X)e^{-\lambda_1(\tau-t)}+(1-F(X))\,e^{-\lambda_0(\tau-t)}\right)u(x(\tau))$$
$$-\Delta(F(\widetilde{X}(\tau;X,t))-F(X))e^{\Delta\tau}e^{\lambda_0 t}\left(\int_\tau^{+\infty}e^{-\lambda_1 s}u(x(s))ds\right)\Bigg)d\tau.$$

Integrating by parts yields

$$\Delta\int_t^{+\infty}(F(\widetilde{X}(\tau;X,t))-F(X))e^{\Delta\tau}e^{\lambda_0 t}\left(\int_\tau^{+\infty}e^{-\lambda_1 s}u(x(s))ds\right)d\tau$$
$$=\Delta\int_t^{+\infty}e^{\lambda_0 t}e^{-\lambda_1\tau}\left(\int_t^{\tau}(F(\widetilde{X}(s;X,t))-F(X))e^{\Delta s}ds\right)u(x(\tau))d\tau$$
$$=-F(X)\int_t^{+\infty}\left(e^{-\lambda_0(\tau-t)}-e^{-\lambda_1(\tau-t)}\right)d\tau+\Delta\int_t^{+\infty}e^{-\lambda_0(\tau-t)}e^{-\Delta\tau}\left(\int_t^{\tau}F(\widetilde{X}(s;X,t))e^{\Delta s}ds\right)u(x(\tau))d\tau$$

Inserting into (C.2) yields

$$\text{(C.3)}\qquad \int_t^{+\infty} e^{-\lambda_0(\tau-t)}\left(1-\Delta e^{-\Delta\tau}\int_t^{\tau} F(\widetilde{X}(s;X,t))e^{\Delta s}ds\right)u(x(\tau))d\tau$$

Changing variables and setting $\tau' = \tau - t$ and $s' = s - t$, we obtain

$$\text{(C.4)}\qquad \int_0^{+\infty} e^{-\lambda_0\tau'}\left(1-\Delta e^{-\Delta\tau'}\int_0^{\tau'} F(\widetilde{X}(s'+t;X,t))e^{\Delta s'}ds'\right)u(x(\tau'+t))d\tau'.$$

Or, relabelling variables for keeping consistency of our notations,

$$\text{(C.5)}\qquad \int_0^{+\infty} e^{-\lambda_0\tau}\left(1-\Delta e^{-\Delta\tau}\int_0^{\tau} F(\widetilde{X}(s+t;X,t))e^{\Delta s}ds\right)u(x(\tau+t))d\tau.$$

Observe that, if the trajectory $\widetilde{X}(\tau+t;X,t)$ with the associated actions $\widetilde{x}(\tau+t;X,t) = \frac{\partial\widetilde{X}}{\partial\tau}(\tau+t;X,t)$ were to maximize the right-hand side of (6.3), the trajectory $\widetilde{X}(\tau;X,0)$ ($\equiv \widetilde{X}(\tau;X)$ with our simplified notations) with the associated actions $\widetilde{x}(\tau;X,0) = \frac{\partial\widetilde{X}}{\partial\tau}(\tau;X,0)$ would also achieve the maximand for (C.5). Finally, inserting into the right-hand side of (6.3) and simplifying, this right-hand side writes as

$$\int_0^{+\infty} e^{-r\tau}\left(1-G(\tau|\mathbf{x}^{\tau},X)\right)u(x(\tau))d\tau.$$

which finally rewrites as (6.4). *Q.E.D.*

Proof of Lemma 4: We rewrite the right-hand side of (6.6) as

$$\text{(C.6)}\qquad \frac{1-G(\tau-t|\mathbf{x}^{\tau-t},\widetilde{X}(t;0))}{1-G(\tau'-t|\mathbf{x}^{\tau'-t},\widetilde{X}(t;0))} = \frac{1-\Delta e^{-\Delta(\tau-t)}\int_0^{\tau-t}F(\widetilde{X}(s,\widetilde{X}(t,0)))e^{\Delta s}ds}{1-\Delta e^{-\Delta(\tau'-t)}\int_0^{\tau'-t}F(\widetilde{X}(s,\widetilde{X}(t,0)))e^{\Delta s}ds}.$$

Changing variables and setting $s' = s - t$, we get

$$1-\Delta e^{-\Delta(\tau-t)}\int_0^{\tau-t}F(\widetilde{X}(s,\widetilde{X}(t,0)))e^{\Delta s}ds = 1-\Delta e^{-\Delta\tau}\int_t^{\tau}F(\widetilde{X}(s'-t,\widetilde{X}(t,0)))e^{\Delta s'}ds'$$

$$= 1-\Delta e^{-\Delta\tau}\int_t^{\tau}F(\widetilde{X}(s',0))e^{\Delta s'}ds'.$$

Let denote by $\chi(t,\tau)$ this last quantity, we notice that

$$\text{(C.7)}\qquad \frac{1-G(\tau-t|\mathbf{x}^{\tau-t},\widetilde{X}(t;0))}{1-G(\tau'-t|\mathbf{x}^{\tau'-t},\widetilde{X}(t;0))} = \frac{\chi(t,\tau)}{\chi(t,\tau')}.$$

Observe now that

$$\frac{\partial}{\partial t}\log(\chi(t,\tau)) = \frac{\frac{\partial\chi}{\partial t}(t,\tau)}{\chi(t,\tau)} = \frac{\Delta e^{\Delta t}F(\widetilde{X}(t,0))}{e^{\Delta\tau}-\Delta\int_t^{\tau}F(\widetilde{X}(s,0))e^{\Delta s}ds}$$

and thus

$$\frac{\partial^2}{\partial\tau\partial t}\log(\chi(t,\tau)) = -\frac{\Delta^2 e^{\Delta t}e^{\Delta\tau}F(\widetilde{X}(t,0))\left(1-F(\widetilde{X}(\tau,0))\right)}{\left(e^{\Delta\tau}-\Delta\int_t^{\tau}F(\widetilde{X}(s,0))e^{\Delta s}ds\right)^2} < 0.$$

From this it follows that, for all $0 < t < \tau < \tau'$, we have

$$\log(\chi(t,\tau')) - \log(\chi(t,\tau)) < \log(\chi(0,\tau')) - \log(\chi(0,\tau))$$

or

$$\frac{\chi(t,\tau)}{\chi(t,\tau')} > \frac{\chi(0,\tau)}{\chi(0,\tau')}.$$

Inserting into (C.7) yields (6.6). *Q.E.D.*

Proof of Proposition 7 : Observe that $\widetilde{Z}(\tau;\mathbf{x}^\tau,X)$ satisfies

$$\text{(C.8)}\qquad \frac{\partial \widetilde{Z}}{\partial \tau}(t;X) = \Delta(1 - F(\widetilde{X}(t\tau;X)) - \widetilde{Z}(\tau;X)) \text{ with } \widetilde{Z}(0;X) = 1.$$

Existence and Properties of $\mathcal{V}^c(X)$. Existence of a solution to the optimization problem (6.4) follows from applying Dmitruk and Kuzkina (2005, Theorem 1, together with the analysis of the special case of discounting in Section 5 of this paper). $\mathcal{V}^c(X) < +\infty$ thus exists. Denote by $(\widetilde{X}^c(\tau;X), \widetilde{Z}^c(\tau;X), x^c(\tau;X))$ an optimal arc. By definition of the suboptimality of the arc $(\widetilde{X}^c(\tau;X'), \widetilde{Z}^c(\tau;X'), x^c(\tau;X'))$ when the initial stock is X', the following inequality holds for any pair (X,X'):

$$\mathcal{V}^c(X) \geq \int_0^{+\infty} \left(1 - \Delta e^{-\Delta\tau} \int_0^\tau F(X + \int_0^\tau x^c(s;X'))e^{\Delta s} ds\right) u(x^c(\tau;X'))d\tau.$$

We express the right-hand side in terms of $\mathcal{V}^c(X')$ to get:

$$\mathcal{V}^c(X) - \mathcal{V}^c(X') \geq$$

$$\Delta \int_0^{+\infty} e^{-\Delta\tau} \int_0^\tau \left(\left(F(X + \int_0^\tau x^c(s;X')) - F(X' + \int_0^\tau x^c(s;X'))\right) e^{\Delta\tau} ds\right) u(x^c(\tau;X'))d\tau.$$

From which, we deduce

$$|\mathcal{V}^c(X') - \mathcal{V}^c(X)| \leq \Delta\mathcal{V}_\infty \|f\| |X' - X|.$$

Hence, $\mathcal{V}^c(X)$ is Lipschitz continuous, and thus absolutely continuous and a.e. differentiable with a derivative given by (C.18) below.

Maximum Principle. We now define the Hamiltonian for this optimization problem as

$$\mathcal{H}(\widetilde{X}, \widetilde{Z}, x, \tau, \mu, \nu) = e^{-\lambda_0\tau}\widetilde{Z}u(x) + \mu x + \nu\Delta(1 - F(\widetilde{X}) - \widetilde{Z})$$

where μ and ν are respectively the costate variables for (4.3) and (C.8). The *Maximum Principle* now gives us the following necessary conditions for optimality of an arc $(\widetilde{X}^c(\tau;X), \widetilde{Z}^c(\tau;X), x^c(\tau;X))$.[42]

Costate variables. $\mu(\tau;X)$ and $\nu(\tau;X)$ are both continuously differentiable on $\mathbb{R}_+$ with

$$-\dot{\mu}(\tau,X) = \frac{\partial \mathcal{H}}{\partial \widetilde{X}}(\widetilde{X}^c(\tau;X), \widetilde{Z}^c(\tau;X), x^c(\tau;X), \mu(\tau;X), \nu(\tau;X))$$

or

$$\text{(C.9)}\qquad \dot{\mu}(\tau;X) = \Delta f(\widetilde{X}^c(\tau;X))\nu(\tau;X) \quad \forall \tau \geq 0;$$

[42]Seierstad and Sydsaeter (1987).

and

$$-\dot{\nu}(\tau, X) = \frac{\partial \mathcal{H}}{\partial \widetilde{Z}}(\widetilde{X}^c(\tau; X), \widetilde{Z}^c(\tau; X), x^c(\tau; X), \mu(\tau; X), \nu(\tau; X))$$

or

$$\dot{\nu}(\tau; X) = -e^{-\lambda_0 \tau} u(x^c(\tau; X)) + \Delta \nu(\tau; X) \quad \forall \tau \geq 0. \tag{C.10}$$

Transversality conditions. The boundary conditions $\widetilde{X}^c(X, 0) = X$ and $\widetilde{Z}^c(X, 0) = 1$ imply that there are no transversality conditions on $\mu(\tau; X)$ and $\nu(\tau; X)$ at $\tau = 0$. Applying Michel (1982, Theorem, p. 997), the necessary transversality condition at $+\infty$ writes as

$$\lim_{\tau \to +\infty} \mathcal{H}(\widetilde{X}^c(\tau; X), \widetilde{Z}^c(\tau; X), x^c(\tau; X), \mu(\tau; X), \nu(\tau; X)) = 0. \tag{C.11}$$

Given that the integrand is non-negative (Michel, 1982, Corollary, p. 997) this condition implies

$$\lim_{\tau \to +\infty} \mu(\tau; X) = \lim_{\tau \to +\infty} \nu(\tau; X) = 0. \tag{C.12}$$

Control variable $x^c(\tau; X)$.

$$x^c(\tau; X) \in \arg\max_{x \geq 0} \mathcal{H}(\widetilde{X}^c(\tau; X), \widetilde{Z}^c(\tau; X), x, \mu(\tau; X), \nu(\tau; X)).$$

Because $\mathcal{H}(\widetilde{X}^c(\tau; X), \widetilde{Z}^c(\tau; X), x, \mu(\tau; X), \nu(\tau; X))$ is strictly concave in x, an interior solution satisfies

$$\frac{\partial \mathcal{H}}{\partial x}(\widetilde{X}^c(\tau; X), \widetilde{Z}^c(\tau; X), x^c(\tau; X), \mu(\tau; X), \nu(\tau; X)) = 0$$

or

$$x^c(\tau; X) = \zeta + e^{\lambda_0 \tau} \frac{\mu(\tau; X)}{\widetilde{Z}^c(\tau; X)}. \tag{C.13}$$

Characterization. Using (C.12) and integrating (C.9) yields

$$\mu(\tau; X) = -\int_{\tau}^{+\infty} \Delta f(\widetilde{X}^c(s; X)) \nu(s; X) ds. \tag{C.14}$$

The solution for (C.10) is of the form

$$\nu(\tau; X) = Ce^{\Delta \tau} - e^{\Delta \tau} \int_{\tau}^{+\infty} e^{-\lambda_1 s} u(x^c(s; X)) ds$$

where C is some integration constant. From (6.2), we have

$$e^{-\Delta \tau} \leq \widetilde{Z}^c(\tau; X) \leq 1 - F(X) + F(X) e^{-\Delta \tau} \quad \forall \tau \geq 0. \tag{C.15}$$

The only possibility for satisfying the transversality condition (C.12) is thus $C = 0$ and we get

$$\nu(\tau; X) = e^{\Delta \tau} \int_{\tau}^{+\infty} e^{-\lambda_1 s} u(x^c(s; X)) ds. \tag{C.16}$$

Inserting into (C.14) yields

$$\text{(C.17)}\quad \mu(\tau;X) = -\int_{\tau}^{+\infty} \Delta f(\widetilde{X}^c(s;X))e^{\Delta s}\left(\int_s^{+\infty} e^{-\lambda_1 s'}u(x^c(s';X))ds'\right)ds.$$

Finally inserting into (C.13), we obtain

$$x^c(\tau;X) = \zeta - \frac{\Delta e^{\lambda_0\tau}}{\widetilde{Z}^c(\tau;X)}\int_{\tau}^{+\infty} f(\widetilde{X}^c(s;X))e^{\Delta s}\left(\int_s^{+\infty} e^{-\lambda_1 s'}u(x^c(s';X))ds'\right)ds.$$

Changing variables yields

$$x^c(\tau;X) = \zeta - \frac{\Delta e^{\lambda_0\tau}}{\widetilde{Z}^c(\tau;X)}\int_0^{+\infty} f(\widetilde{X}^c(s'+\tau;X))e^{\Delta(s'+\tau)}\left(\int_{s'+\tau}^{+\infty} e^{-\lambda_1 s''}u(x^c(s'';X))ds''\right)ds'$$

and thus

$$x^c(\tau;X) = \zeta - \frac{\Delta e^{\lambda_0\tau}}{\widetilde{Z}^c(\tau;X)}\int_0^{+\infty} f(\widetilde{X}^c(s'+\tau;X))e^{\Delta(s'+\tau)}\left(\int_{s'}^{+\infty} e^{-\lambda_1 (s'''+\tau)}u(x^c(s'''+\tau;X))ds'''\right)ds'.$$

We thus rewrite this condition as (6.7).

In particular, at any point of differentiability of $\mathcal{V}^c(X)$, we have

$$\text{(C.18)}\quad \dot{\mathcal{V}}^c(X) = \mu(0;X).$$

Inserting into (C.13) gives (6.10). *Q.E.D.*

Proof of Proposition 8: By definition, we have

$$\widetilde{X}^c(t';X) = \widetilde{X}^c(t;X) + \int_t^{t'} x^c(\tau';X)d\tau' \quad \forall X, \forall t' \geq t \geq 0.$$

If (6.11) were to hold, we would have

$$\widetilde{X}^c(t';X) = \widetilde{X}^c(t;X) + \int_t^{t'} x^c(\tau'-t;\widetilde{X}^c(t;X))d\tau' \quad \forall X, \forall t' \geq t \geq 0.$$

Changing variables in the integral yields

$$\widetilde{X}^c(t';X) = \widetilde{X}^c(t;X) + \int_0^{t'-t} x^c(\tau;\widetilde{X}^c(t;X))d\tau \quad \forall X, \forall t' \geq t \geq 0$$

and thus, by direct integration,

$$\text{(C.19)}\quad \widetilde{X}^c(t';X) = \widetilde{X}^c(t'-t;\widetilde{X}^c(t;X)) \quad \forall X, \forall t' \geq t \geq 0.$$

By simply adapting definition (6.7), re-optimizing at date $t > 0$ when the stock has reached level $\widetilde{X}^c(t;X)$ yields an optimal action for date $t' \geq t$ which is now given by

$$x^c(t'-t;\widetilde{X}^c(t;X)) = \zeta -$$

$$\frac{\Delta}{\widetilde{Z}^c(t'-t;\widetilde{X}^c(t;X))}\int_0^{+\infty} e^{-\lambda_1\tau}\left(\int_0^{\tau} f(\widetilde{X}^c(t'-t+s;\widetilde{X}^c(t;X)))e^{\Delta s}ds\right)u(x^c(t'-t+\tau;\widetilde{X}^c(t;X)))d\tau, \forall t' \geq t$$

where

$$\widetilde{Z}^c(t'-t;\widetilde{X}^c(t;X)) = 1-\Delta e^{-\Delta(t'-t)}\int_0^{t'-t} F(\widetilde{X}^c(t'-t+\tau;\widetilde{X}^c(t;X)))e^{\Delta\tau}d\tau.$$

In particular, taking $t'=t$ yields

(C.20)

$$x^c(0;\widetilde{X}^c(t;X)) = \zeta - \Delta\int_0^{+\infty} e^{-\lambda_1\tau}\left(\int_0^{\tau} f(\widetilde{X}^c(s;\widetilde{X}^c(t;X)))e^{\Delta s}d\tau\right)u(x^c(\tau;\widetilde{X}^c(t;X)))d\tau, \forall t' \geq t.$$

Using (6.11) and (C.19), we simplify (C.20) as

(C.21)

$$x^c(0;\widetilde{X}^c(t;X)) = \zeta - \Delta\int_0^{+\infty} e^{-\lambda_1\tau}\left(\int_0^{\tau} f(\widetilde{X}^c(t+s;X))e^{\Delta s}d\tau\right)u(x^c(t+\tau;X))d\tau \quad \forall t\geq 0.$$

If the solution is time-consistent, (6.11) would imply that the expressions in (6.7) and (C.21) are the same. However, it can never be possible for $t>0$ since then $\widetilde{Z}^c(t;X)<1$. *Q.E.D.*

Extended Value Function

The expression of $\mathcal{V}^c(X)$ given in (6.4) suggests that the state of the system is best described by adding to the value of the current stock X another state variable that reflects how the probability of survival evolves in the future. Two trajectories that reach the same value for the current stock at date t and keep the same survival rate should be optimally continued the same way. Instead, two trajectories that have reached the same stock of past actions but are thought to survive with different probabilities might be pursued along two different paths.

To expand state variables and restore the force of dynamic programming, we now use the *survival index* $\widetilde{Z}$ as another state variable. To be able to track how this variable evolves, we consider the following law of motion

$$\text{(C.22)} \quad \frac{\partial\widetilde{Z}}{\partial\tau}(\tau;X,Z) = \Delta(1-F(\widetilde{X}(\tau;X,Z))-\widetilde{Z}(\tau;X,Z)) \text{ with } \widetilde{Z}(0;X,Z)=Z$$

where we now make explicit the dependence of the stock $\widetilde{X}(\tau;X,Z)$ on the new state variable Z. As the stock $\widetilde{X}(\tau;X,Z)$ increases, the term $1-F(\widetilde{X}(\tau;X,Z))$, that represents the probability that the tipping point lies above the existing stock, decreases and DM becomes less optimistic. On the other hand, once $\widetilde{Z}(\tau;X,Z)$ decreases, surviving is viewed as being relatively good news, an effect which pushes the survival index up.

Integrating (C.22), we immediately get

$$\text{(C.23)} \quad \widetilde{Z}(\tau;X,Z) = (Z-1)e^{-\Delta\tau}+1-\Delta e^{-\Delta\tau}\int_0^{\tau} F(\widetilde{X}(s;X,Z))e^{\Delta s}ds.$$

From this, we may define the *extended value function* $\mathcal{V}_e^c(X,Z)$ for any $X\geq 0$ and any $Z\in(0,1]$ as

$$\text{(C.24)} \quad Z\mathcal{V}_e^c(X,Z) = \sup_{\mathbf{x},\widetilde{X}(\cdot),\widetilde{Z}(\cdot) \text{ s.t. (4.3) and (C.22)}} \int_0^{+\infty} e^{-\lambda_0\tau}\widetilde{Z}(\tau;X,Z)u(x(\tau))d\tau.$$

The choice to factorize Z on the left-hand side views the *extended value function* as being computed conditionally on survival up to that point. Together with an associated *feedback rule* $\sigma_e^c(X,Z)$, this extended value function defines the full trajectory of the system both in terms of the overall stock $\widetilde{X}(\tau;X,Z)$ but also of the survival index $\widetilde{Z}(\tau;X,Z)$. In particular, the intertemporal payoff $\mathcal{V}^c(0)=\mathcal{V}_e^c(0,1)$ is achieved by adopting the action profile $\sigma^c(\widetilde{X}(\tau;0,1),\widetilde{Z}(\tau;0,1))$ starting from the initial conditions of the system $X=0$ and $Z=1$.

The extended value function $\mathcal{V}_e^c(X,Z)$ is a mere technical device to compute the value function $\mathcal{V}^c(X)$. It allows to use dynamic programming techniques in a time-inconsistency context.[43] Yet, the feedback rule $\sigma_e^c(X,Z)$ cannot be viewed as a practical way of guiding actions. Indeed, obeying to such a rule requires to keep track of beliefs along the trajectory. It might be a task of tall order, especially when beliefs might be manipulated; an issue of prime importance in a context where experts may face difficulties in conveying evidence. It is thus natural to look at a solution concept that does not depend on the survival index; a task undertaken in Section 7.

Next proposition provides some key properties of the extended value function $\mathcal{V}_e^c(X,Z)$.

PROPOSITION C.1 *A continuously differentiable extended value function $\mathcal{V}_e^c(X,Z)$ satisfies the following bi-dimensional HBJ partial differential equation:*

$$\text{(C.25)}\quad \frac{\partial \mathcal{V}_e^c}{\partial X}(X,Z)=-\zeta+\sqrt{\zeta^2+2\left(\lambda_0-\frac{\Delta(1-F(X)-Z)}{Z}\right)\mathcal{V}_e^c(X,Z)-2\lambda_1\mathcal{V}_\infty-2\Delta(1-F(X)-Z)\frac{\partial \mathcal{V}_e^c}{\partial Z}(X,Z)}$$

with the boundary condition

$$\text{(C.26)}\quad 0\le Z\mathcal{V}_e^c(X,Z)\le\left(F(X)+(1-F(X))\frac{\lambda_1}{\lambda_0}\right)\mathcal{V}_\infty\quad \forall X\ge 0,\forall Z\in(0,1].$$

The feedback rule is

$$\text{(C.27)}\quad \sigma_e^c(X,Z)=\zeta+\frac{\partial \mathcal{V}_e^c}{\partial X}(X,Z).$$

Confirming earlier remarks, the structure of payoffs as in (C.24) shows that the discount rate for date τ payoffs is now *non-constant* and of the form

$$\mathcal{R}(\tau,X,Z)=\lambda_0-\frac{\dot{\widetilde{Z}}(\tau;X,Z)}{\widetilde{Z}(\tau;X,Z)}\equiv\lambda_0-\frac{\Delta(1-F(\widetilde{X}(\tau;X,Z))-\widetilde{Z}(\tau;X,Z))}{\widetilde{Z}(\tau;X,Z)}$$

Using the survival index as a state variable allows to keep track of this time-dependency. Future payoffs are counted with this *non-constant* discounting, which explains the factor of $\mathcal{V}_e^c(X,Z)$ on the right-hand side of (C.25). The choice of an action $x(t)$ at date t has no direct consequences on how this implicit discount rate evolves, as it can be seen on (C.22). Yet, because the stock and

[43] Marcet and Marimon (2019) have presented a general theory of discrete-time optimization problems with forward-looking constraints, a feature that prevails in a number of macroeconomic and political economy contexts (Aiyagari et al., 2002, Acemoglu et al., 2011, Attanasio and Rios-Rull, 2002 among others). Our continuous time model is somewhat simpler since, in the scenario of deep uncertainty, payoffs themselves have a forward-looking component. Marcet and Marimon (2019) have shown how to recover a recursive structure to the optimization problems by adding multipliers of the forward-looking constraints as state variables which follow a specific evolution. In our context, a recursive structure can be found when the belief index is used as an extra state variable

the survival index themselves evolve over time, the value of this discount rate keeps changing and DM must take this into account to assess how his future payoffs will vary with the index. It explains the last term on the the right-hand side of (C.25).

PROOF OF PROPOSITION C.1: We first slightly abuse notations and vocabulary to define another value function $\mathcal{W}_e^c(X, Z)$ as

$$\mathcal{W}_e^c(X, Z) = Z\mathcal{V}_e^c(X, Z).$$

PROPOSITION C.2 *A continuously differentiable value function $\mathcal{W}_e^c(X, Z)$ satisfies the following HBJ equation:*

$$\lambda_0 \mathcal{W}_e^c(X, Z) = \lambda_1 \mathcal{V}_\infty Z + \zeta \frac{\partial \mathcal{W}_e^c}{\partial X}(X, Z) + \frac{1}{2Z}\left(\frac{\partial \mathcal{W}_e^c}{\partial X}(X, Z)\right)^2 + \Delta(1 - F(X) - Z)\frac{\partial \mathcal{W}_e^c}{\partial Z}(X, Z) \tag{C.28}$$

with the boundary conditions

$$0 \leq \mathcal{W}_e^c(X, Z) \leq \mathcal{V}_\infty + (1 - F(X))\frac{\Delta}{\lambda_0}\mathcal{V}_\infty. \tag{C.29}$$

The feedback rule is given by

$$\sigma_e^c(X, Z) = \zeta + \frac{1}{Z}\frac{\partial \mathcal{W}_e^c}{\partial X}(X, Z). \tag{C.30}$$

PROOF OF PROPOSITION C.2: Consider $Z \in [0, 1]$. Using the *Dynamic Programming Principle*, $\mathcal{W}_e^c(X, Z)$ satisfies

$$\mathcal{W}_e^c(X, Z) = \sup_{\mathbf{x}, \widetilde{X}(\cdot), \widetilde{Z}(\cdot) \text{ s.t. (4.3) and (C.22)}} \int_0^\varepsilon e^{-\lambda_0 t}\widetilde{Z}(t; X, Z)u(x(t))dt \tag{C.31}$$

$$+ e^{-\lambda_0 \varepsilon}\mathcal{W}^c(\widetilde{X}(\varepsilon; X, Z), \widetilde{Z}(\varepsilon; X, Z)).$$

Consider now ε small enough and denote by x a fixed action over the interval $[0, \varepsilon]$. From (4.3) and (C.22), we get

$$\widetilde{X}(\varepsilon; X, Z) = X + \varepsilon x + o(\varepsilon)$$

and

$$\widetilde{Z}(\varepsilon; X, Z) = Z + \varepsilon\Delta(1 - F(X) - Z) + o(\varepsilon)$$

where $\lim_{\varepsilon \to 0} o(\varepsilon)/\varepsilon = 0$.

When $\mathcal{W}_e^c(X, Z)$ is continuously differentiable, we can take a first-order Taylor expansion in ε of the maximand in (C.31) to write it as:

$$\mathcal{W}_e^c(X, Z) +$$

$$\varepsilon\left(Zu(x) + x\frac{\partial \mathcal{W}_e^c}{\partial X}(X, Z) + \Delta(1 - F(X) - Z)\frac{\partial \mathcal{W}_e^c}{\partial Z}(X, Z) - \lambda_0 \mathcal{W}_e^c(X, Z)\right) + o(\varepsilon).$$

Inserting into (C.31) yields the following HBJ equation:

$$\lambda_0 \mathcal{W}_e^c(X,Z) = \sup_{x\in\mathcal{X}} \left\{ Zu(x) + x\frac{\partial \mathcal{W}_e^c}{\partial X}(X,Z) + \Delta(1-F(X)-Z)\frac{\partial \mathcal{W}_e^c}{\partial Z}(X,Z)\right\}. \tag{C.32}$$

Feedback Rule. From this, it immediately follows that the feedback rule is given by (C.30). Simplifying (C.32) using the feedback rule (C.30) finally yields (C.28).

Boundary Condition. Now, observe that (C.22) and $F(X) \leq F(\widetilde{X}(\tau;X,Z)) \leq 1$ imply

$$0 \leq \frac{\partial}{\partial \tau}\left(\widetilde{Z}(\tau;X,Z)e^{\Delta\tau}\right) \leq \Delta(1-F(X))e^{\Delta\tau}.$$

Integrating between 0 and t yields

$$0 \leq Ze^{-\Delta\tau} \leq \widetilde{Z}(\tau;X,Z) \leq Ze^{-\Delta\tau} + (1-F(X))\left(1-e^{-\Delta\tau}\right).$$

From this and the fact that $0 \leq Z \leq 1$, it follows that

$$0 \leq Ze^{-\Delta\tau} \leq \widetilde{Z}(\tau;X,Z) \leq F(X)e^{-\Delta\tau} + 1 - F(X) \leq 1. \tag{C.33}$$

Henceforth, the whole trajectory $\widetilde{Z}(\tau;X,Z)$ always remains in the stable domain $[0,1]$.

From the third inequality in (C.33), taking maximum on the right-hand side of (C.24), the right-hand side inequality of (C.29) follows. From the first inequality in (C.33), we immediately get the left-hand side inequality of (C.29). *Q.E.D.*

A Verification Theorem. Proposition C.3 below shows that the conditions given Proposition C.1 to characterize the commitment value function by means of an HBJ equation together with boundary conditions are in fact sufficient. We again follow Ekeland and Turnbull (1983, Theorem 1, p. 6) to derive a *Verification Theorem.*

Proposition C.3 *Assume first that there exists a continuously differentiable function $\mathcal{W}_0(X,Z)$ which satisfies:*

(C.34)

$$\lambda_0\mathcal{W}_0(X,Z) \geq \widetilde{Z}(t;X,Z)u(x) + x\frac{\partial \mathcal{W}_0}{\partial X}(X,Z) + \Delta(1-F(X)-\widetilde{Z}(t;X,Z))\frac{\partial \mathcal{W}_0}{\partial Z}(X,Z) \quad \forall (x,X,Z);$$

and, second, that there exists an action profile $\mathbf{x}_0$ and a path $X_0(t) = \int_0^t x_0(\tau)d\tau$, $Z_0(t) = 1-\Delta e^{-\Delta t}\int_0^t F(X_0(\tau))e^{\Delta\tau}d\tau$ such that

$$\lambda_0\mathcal{W}_0(X_0(t),Z_0(t)) = Z_0(t)u(x_0(t)) \tag{C.35}$$

$$+x_0(t)\frac{\partial \mathcal{W}_0}{\partial X}(X_0(t),Z_0(t)) + \Delta(1-F(X_0(t))-Z_0(t))\frac{\partial \mathcal{W}_0}{\partial Z}(X_0(t),Z_0(t)) \quad \forall t \geq 0.$$

Then $\mathbf{x}_0$ is an optimal action profile with its associated path $(X_0(t),Z_0(t))$.

PROOF OF PROPOSITION C.3: Suppose thus that $\mathcal{W}_e^c(X,Z)$ as characterized in Proposition C.2 is continuously differentiable. It is our candidate for the function $\mathcal{W}_0(X,Z)$ in the statement of Proposition C.3. By definition (C.32), we have

$$\lambda_0 \mathcal{W}_e^c(X,Z) = Zu(\sigma_e^c(X,Z)) + \sigma_e^c(X,Z)\frac{\partial \mathcal{W}_e^c}{\partial X}(X,Z) + \Delta(1-F(X)-Z)\frac{\partial \mathcal{W}_e^c}{\partial Z}(X,Z), \quad \forall (X,Z)$$

and thus

$$\lambda_0 \mathcal{W}_e^c(X,Z) \geq Zu(x) + x\frac{\partial \mathcal{W}_e^c}{\partial X}(X,Z) + \Delta(1-F(X)-Z)\frac{\partial \mathcal{W}_e^c}{\partial Z}(X,Z), \quad \forall (x,X,Z) \tag{C.36}$$

where the inequality comes from the fact that $\sigma_e^c(X,Z)$ maximizes the right-hand side.

To get (C.35), we use again (C.32) but now applied to the path $(x^c(t), X^c(t), Z^c(t))$ where $X^c(t)$ is such that $\dot{X}^c(t) = x^c(t) = \sigma^c(X^c(t), Z^c(t))$ with $X^c(0) = 0$ and $Z^c(t) = 1 - \Delta e^{-\Delta t}\int_0^t F(X^c(\tau))e^{\Delta \tau}d\tau$.

Define now a value function $\widetilde{\mathcal{W}^c}(X,Z,t) = e^{-\lambda_0 t}\mathcal{W}_e^c(X,Z)$. By (C.36), we get

$$0 \geq \frac{\partial \widetilde{\mathcal{W}^c}}{\partial t}(X,Z,t) + x\frac{\partial \widetilde{\mathcal{W}^c}}{\partial X}(X,Z,t) + \Delta(1-F(X)-Z)\frac{\partial \widetilde{\mathcal{W}^c}}{\partial Z}(X,Z,t) + e^{-\lambda_0 t}Zu(x) \quad \forall (x,X,Z). \tag{C.37}$$

Using $x^c(t) = \sigma^c(X^c(t), Z^c(t))$, $Z^c(t) = 1 - \Delta e^{-\Delta t}\int_0^t F(X^c(\tau))e^{\Delta\tau}d\tau$ and (B.14), we also get

$$\begin{aligned} 0 = {} & \frac{\partial \widetilde{\mathcal{W}^c}}{\partial t}(X^c(t), Z^c(t), t) + x^c(t)\frac{\partial \widetilde{\mathcal{W}^c}}{\partial X}(X^c(t), Z^c(t), t) \\ & + \Delta(1 - F(X^c(t)) - Z^c(t))\frac{\partial \widetilde{\mathcal{W}^c}}{\partial Z}(X^c(t), Z^c(t), t) + e^{-\lambda_0 t}Z^c(t)u(x^c(t)) \quad \forall t \geq 0. \end{aligned} \tag{C.38}$$

Take now an arbitrary action plan $\mathbf{x}$ with the associated path $X(t) = \int_0^t x(\tau)d\tau$ and $Z(t) = 1 - \Delta e^{-\Delta t}\int_0^t F(X(\tau))e^{\Delta\tau}d\tau$. Let us fix an arbitrary $T > 0$. Integrating (C.37) along the path $(x(t), X(t), Z(t))$, we compute

$$\begin{aligned} 0 \geq \int_0^T \Bigg(& \frac{\partial \widetilde{\mathcal{W}^c}}{\partial t}(X(t), Z(t), t) + x(t)\frac{\partial \widetilde{\mathcal{W}^c}}{\partial X}(X(t), Z(t), t) \\ & + \Delta(1 - F(X(t)) - Z(t))\frac{\partial \widetilde{\mathcal{W}^c}}{\partial Z}(X(t), Z(t), t) + e^{-\lambda_0 t}Z(t)u(x(t)) \Bigg) dt \end{aligned}$$

or

$$0 \geq \int_0^T \left(\frac{d\widetilde{\mathcal{W}^c}}{dt}(X(t), Z(t), t) + e^{-\lambda_0 t}Z(t)u(x(t)) \right) dt \quad \forall T \geq 0.$$

By definition of the total derivative of $\widetilde{\mathcal{W}^c}(X(t), Z(t), t)$ with respect to time, we thus get

$$\widetilde{\mathcal{W}^c}(0,0,0) \geq \widetilde{\mathcal{W}^c}(X(T), Z(T), T) + \int_0^T e^{-\lambda_0 t}Z(t)u(x(t))dt \quad \forall T \geq 0.$$

Because $\widetilde{\mathcal{W}^c}(X,Z,t) = e^{-\lambda_0 t}\mathcal{W}^c(X,Z) \geq 0$ for all (X,Z,t), we obtain:

$$\mathcal{W}^c(0,0) \geq e^{-\lambda_0 T}\mathcal{W}^c(X(T), Z(T)) + \int_0^T e^{-\lambda_0 t}Z(t)u(x(t))dt \quad \forall T \geq 0.$$

Because of the boundary conditions (C.29), $e^{-\lambda_0 T}\mathcal{W}^c(X(T),Z(T))$ converges towards zero as $T\to+\infty$ for any feasible path. Moreover, for any such feasible path $\int_0^{+\infty} e^{-\lambda_0 t}Z(t)u(x(t))dt$ exists. Henceforth, we get:

$$\mathcal{W}^c(0,0)\geq \sup_{\mathbf{x}}\int_0^{+\infty} e^{-\lambda_0 t}Z(t)u(x(t))dt$$

which shows that $(x^c(t),X^c(t),Z^c(t))$ is indeed an optimal path. *Q.E.D.*

Boundary condition. An immediate corollary of (C.29) is thus (C.26).

Partial Differential Equation. Rewriting the optimality conditions in terms of $\mathcal{V}_e^c(X,Z)$, (C.28) becomes

$$\lambda_0\mathcal{V}_e^c(X,Z)=\lambda_1\mathcal{V}_\infty+\zeta\frac{\partial\mathcal{V}_e^c}{\partial X}(X,Z)+\frac{1}{2}\left(\frac{\partial\mathcal{V}_e^c}{\partial X}(X,Z)\right)^2+\frac{\Delta(1-F(X)-Z)}{Z}\frac{\partial\mathcal{W}_e^c}{\partial Z}(X,Z).$$

Solving this second-degree equation and keeping solution ensuring a positive feedback rule yields

$$\text{(C.39)}\quad \frac{\partial\mathcal{V}^c}{\partial X}(X,Z)=-\zeta+$$
$$\sqrt{\zeta^2+2\lambda_0\mathcal{V}_e^c(X,Z)-2\lambda_1\mathcal{V}_\infty-2\frac{\Delta(1-F(X)-Z)}{Z}\frac{\partial}{\partial Z}(Z\mathcal{V}_e^c(X,Z))}.$$

Developing further yields (C.25).

Q.E.D.

Long-Run Behavior

We now characterize the asymptotic behavior of the commitment solution. In the long run, the optimal commitment solution entails almost choosing a fix action $\sigma^c\in[0,\zeta]$. This action solves

$$\text{(C.40)}\quad \frac{(\zeta-\sigma^c)(\lambda_0+\dot{R}(0)\sigma^c)}{\Delta-\dot{R}(0)\sigma^c}=\dot{R}(0)\left(\mathcal{V}_\infty-\frac{1}{2\lambda_1}(\zeta-\sigma^c)^2\right).$$

Observe that the right-hand side of (C.40) is worth $\dot{R}(0)\mathcal{V}_\infty>0$ at $\sigma^c=\zeta$ while the left-hand side is worth zero. The left-hand side of (C.40) is worth $\frac{\zeta\lambda_0}{\Delta}$ at $\sigma^c=0$ while the right-hand side is $\dot{R}(0)\frac{D}{2\lambda_1}$. Henceforth, there exists a solution σ^c to (C.40) when

$$\text{(C.41)}\quad \frac{\zeta\lambda_0}{\Delta}\geq\frac{\dot{R}(0)D}{2\lambda_1}.$$

Notice that the right-hand side of (C.40) is an increasing function of σ^c while the left-hand side is decreasing when

$$\text{(C.42)}\quad \frac{\lambda_0\Delta}{\lambda_1}>\dot{R}(0)\zeta.$$

Under those conditions, there exists a unique solution to (C.40) that belongs to $[0,\zeta]$. Next proposition completes the description of such long-run behavior.

Proposition C.4 *The optimal action, stock and belief index admit the following approximations when t is large:*

(C.43) $x^c(t;X) \sim_{+\infty} \sigma^c,$

(C.44) $\widetilde{X}^c(t;X) \sim_{+\infty} \sigma^c \tau,$

(C.45) $\widetilde{Z}^c(t;X) \sim_{+\infty} \frac{\Delta}{\Delta - k\sigma^c}.$

One important point is that the limiting behavior of the solution is independent of the initial condition X. In other words, DM's incentives to modify the trajectory as time goes somewhat vanish in the long run. The time-inconsistency problem is a problem in the short run only.

Proof of Proposition C.4: To look at the long-run behavior of the commitment solution, we first change variables and define

$$Y(\tau;X) = 1 - F(\widetilde{X}^c(\tau;X)) \in [0,1]\,, R(Y(\tau;X)) = f(F^{-1}(1 - Y(\tau;X))),$$

$$\tilde{\nu}(\tau;X) = \nu(\tau;X)e^{\lambda_0 \tau}, \tilde{\mu}(\tau;X) = \mu(\tau;X)e^{\lambda_0 \tau}.$$

Using (C.9) and (C.10), we rewrite

(C.46) $\dot{\tilde{\mu}}(\tau;X) = \lambda_0 \tilde{\mu}(\tau;X) + \Delta R(Y(\tau;X))\tilde{\nu}(\tau;X) \quad \forall \tau \geq 0;$

and

(C.47) $\dot{\tilde{\nu}}(\tau;X) = \lambda_1 \left(\tilde{\nu}(\tau;X) - \mathcal{V}_\infty + \frac{1}{2\lambda_1}(\zeta - x^c(\tau;X))^2 \right).$

Inserting into (C.13) yields

(C.48) $x^c(\tau;X) = \zeta + \frac{\tilde{\mu}(\tau;X)}{\widetilde{Z}^c(\tau;X)}$

where

(C.49) $\dot{\widetilde{Z}}^c(\tau;X) = \Delta(Y(\tau;X) - \widetilde{Z}^c(\tau;X)).$

We are looking for a solution which in the neighborhood of $Y(\tau;X) = 0$ (i.e. for $\tau \to +\infty$) that has a linear approximation with two parameters (z, σ^c) such that

(C.50) $\widetilde{Z}^c(\tau;X) \sim_{\tau\to+\infty} zY(\tau;X), \quad \widetilde{x}^c(\tau;X) \sim_{\tau\to+\infty} \sigma^c, \quad \tilde{\nu}(\tau;X) \sim_{\tau\to+\infty} \nu^c.$

In the neighborhood of $\tau \to +\infty$, (C.49) becomes

(C.51) $z\dot{Y}(\tau;X) = \Delta(1-z)Y(\tau;X).$

Differentiating $Y(\tau;X) = 1 - F(\widetilde{X}^c(\tau;X))$ yields in that same neighborhood

(C.52) $\dot{Y}(\tau;X) = -\sigma^c \dot{R}(0) Y(\tau;X).$

Inserting into (6.7) yields

$$z = \frac{\Delta}{\Delta - \dot{R}(0)\sigma^c} > 0 \tag{C.53}$$

where the right-hand side inequality follows from $\Delta > \dot{R}(0)\zeta$ which is itself implied by the second condition in (C.41).

Now, inserting (C.50) into (C.48) yields

$$\sigma^c = \zeta + \frac{\tilde{\mu}(\tau; X)}{zY(\tau; X)}. \tag{C.54}$$

Inserting (C.54) into (C.46), we obtain

$$(\sigma^c - \zeta)z\dot{Y}(\tau; X) = \lambda_0(\sigma^c - \zeta)zY(\tau; X) + \Delta\dot{R}(0)Y(\tau; X)\tilde{\nu}(\tau; X) \quad \forall\tau \geq 0. \tag{C.55}$$

$$-z(\sigma^c - \zeta)\sigma^c\dot{R}(0)Y(\tau; X) = \lambda_0(\sigma^c - \zeta)zY(\tau; X) + \Delta\dot{R}(0)Y(\tau; X)\tilde{\nu}(\tau; X) \quad \forall\tau \geq 0. \tag{C.56}$$

Inserting (C.54) into (C.47), we obtain in the neighborhood of $\tau \to +\infty$

$$\dot{\tilde{\nu}}(\tau; X) = \lambda_1\left(\tilde{\nu}(\tau; X) - \mathcal{V}_\infty + \frac{1}{2\lambda_1}(\zeta - \sigma^c)^2\right). \tag{C.57}$$

The only bounded solution is such that

$$\tilde{\nu}(\tau; X) \sim_{\tau\to+\infty} \nu^c = \mathcal{V}_\infty - \frac{1}{2\lambda_1}(\zeta - \sigma^c)^2. \tag{C.58}$$

Inserting into (C.56) and using (C.53) yields (C.40).

Q.E.D.

APPENDIX D: DEEP UNCERTAINTY, *PSEUDO*-VALUE FUNCTION AND STOCK-MARKOV EQUILIBRIUM

For further reference, we now state the following Lemmatas.

LEMMA D.1

$$\frac{\partial X^*}{\partial X}(\tau; X) = \frac{\sigma^*(X^*(\tau; X))}{\sigma^*(X)}. \tag{D.1}$$

PROOF OF LEMMA D.1: Starting with the definition of $X^*(\tau; X)$ we get:

$$\frac{\partial X^*}{\partial \tau}(\tau; X) = \sigma^*(X^*(\tau; X))$$

and

$$\frac{\partial X^*}{\partial \tau}(\tau; X + dX) = \sigma^*(X^*(\tau; X + dX)).$$

Taking dX small and using a first-order Taylor approximation, we get:

$$\sigma^*(X^*(\tau; X+dX)) = \sigma^*(X^*(\tau; X)) + \dot{\sigma}^*(X^*(\tau; X))\frac{\partial X^*}{\partial X}(\tau; X)dX + o(dX)$$

where $\lim_{dX \to 0} o(dX)/X = 0$. Therefore, we get:

$$\frac{\partial X^*}{\partial \tau}(\tau; X+dX) - \frac{\partial X^*}{\partial \tau}(\tau; X) = \dot{\sigma}^*(X^*(\tau; X))\frac{\partial X^*}{\partial X}(\tau; X)dX + o(dX).$$

Using a first-order Taylor approximation of the left-hand side and simplifying, we get:

$$\frac{\partial}{\partial \tau}\left(\frac{\partial X^*}{\partial X}(\tau; X)\right) = \dot{\sigma}^*(X^*(\tau; X))\frac{\partial X^*}{\partial X}(\tau; X).$$

Thus,

$$\frac{\partial}{\partial \tau}\log\left(\frac{\partial X^*}{\partial X}(\tau; X)\right) = \dot{\sigma}^*(X^*(\tau; X)).$$

Integrating and taking into account that $X^*(0; X) = X$ yields

$$\text{(D.2)} \qquad \frac{\partial X^*}{\partial X}(\tau; X) = exp\left(\int_0^\tau \dot{\sigma}^*(X^*(s; X))ds\right).$$

Using the stationarity of the feedback rule and differentiating with respect to t yields

$$\text{(D.3)} \qquad \dot{\sigma}^*(X^*(\tau; X)) = \frac{\frac{\partial^2 X^*}{\partial \tau^2}(\tau; X)}{\frac{\partial X^*}{\partial \tau}(\tau; X)}.$$

Inserting into (D.2) and integrating yields

$$\frac{\partial X^*}{\partial X}(\tau; X) = exp\left(ln\left(\frac{\frac{\partial X^*}{\partial \tau}(\tau; X)}{\frac{\partial X^*}{\partial \tau}(0; X)}\right)\right)$$

and thus

$$\frac{\partial X^*}{\partial X}(\tau; X) = \frac{\sigma^*(X^*(\tau; X))}{\sigma^*(X^*(0; X))}.$$

Noticing that $X^*(0; X) = X$ yields (D.1). *Q.E.D.*

LEMMA D.2

$$\text{(D.4)} \qquad \frac{\partial \widetilde{X}}{\partial \varepsilon}(x, \varepsilon, \tau; X)|_{\varepsilon=0} = \sigma^*(X^*(\tau; X))\left(\frac{x}{\sigma^*(X)} - 1\right) = \left(\frac{x}{\sigma^*(X)} - 1\right)\frac{\partial \widetilde{X}}{\partial \tau}(\tau; X).$$

PROOF OF LEMMA D.2: Take $\tau > \varepsilon$, we have

$$\widetilde{X}(x, \varepsilon, \tau; X) = X + x\varepsilon + \int_\varepsilon^\tau \sigma^*(\widetilde{X}(x, \varepsilon, s; X))ds$$

Now observe that, for $s \geq \varepsilon$, we have

$$\widetilde{X}(x, \varepsilon, s; X) = X^*(s - \varepsilon, X + x\varepsilon).$$

Hence, we rewrite

$$\widetilde{X}(x,\varepsilon,\tau;X) = X + x\varepsilon + \int_{\varepsilon}^{\tau} \sigma^*(X^*(s-\varepsilon, X + x\varepsilon))ds. \tag{D.5}$$

Differentiating with respect to ε yields

$$\frac{\partial \widetilde{X}}{\partial \varepsilon}(x,\varepsilon,\tau;X)|_{\varepsilon=0} = x - \sigma^*(X) + \int_0^{\tau} \dot{\sigma}^*(X^*(s;X))\left(-\frac{\partial X^*}{\partial s}(s;X) + x\frac{\partial X^*}{\partial X}(s;X)\right)ds. \tag{D.6}$$

Inserting (D.1) into (D.6) yields

$$\frac{\partial \widetilde{X}}{\partial \varepsilon}(x,\varepsilon,\tau;X)|_{\varepsilon=0} = x - \sigma^*(X) + \left(\frac{x}{\sigma^*(X)} - 1\right)\int_0^{\tau} \dot{\sigma}^*(X^*(s;X))\frac{\partial X^*}{\partial s}(s;X)ds.$$

Integrating the last term yields

$$\frac{\partial \widetilde{X}}{\partial \varepsilon}(x,\varepsilon,\tau;X)|_{\varepsilon=0} = x - \sigma^*(X) + \left(\frac{x}{\sigma^*(X)} - 1\right)\left(\sigma^*(X^*(\tau,X)) - \sigma^*(X)\right). \tag{D.7}$$

Simplifying further yields (D.4). *Q.E.D.*

Next Lemma provides a characterization of any continuously differentiable *Stock-Markov Equilibrium* $(\mathcal{V}^*(X), \sigma^*(X)))$.

LEMMA D.3 *If the pseudo-value function $\mathcal{V}^*(X)$ is continuously differentiable, the following necessary conditions hold at a Stock-Markov Equilibrium $(\mathcal{V}^*(X), \sigma^*(X))$:*

$$0 = \max_{x\in\mathcal{X}} \frac{\partial \mathcal{V}}{\partial \varepsilon}(x,0,X), \tag{D.8}$$

$$\sigma^*(X) \in \arg\max_{x\in\mathcal{X}} \frac{\partial \mathcal{V}}{\partial \varepsilon}(x,0,X). \tag{D.9}$$

PROOF OF LEMMA D.3: If the *pseudo*-value function $\mathcal{V}^*(X)$ is continuously differentiable, $\mathcal{V}(x,\varepsilon;X)$ is itself continuously differentiable in ε, and a first-order Taylor expansion in ε yields

$$\mathcal{V}(x,\varepsilon;X) = \mathcal{V}^*(X) + \varepsilon\frac{\partial \mathcal{V}}{\partial \varepsilon}(x,0,X) + o(\varepsilon). \tag{D.10}$$

Hence, (7.8) amounts to (D.8). Conjectures being correct at equilibrium, (D.9) must also hold. *Q.E.D.*

PROOF OF PROPOSITION 9:

LEMMA D.4 *$\mathcal{V}^*(X)$ and $\varphi(X)$ satisfy the following system of first-order differential equations:*

$$\sigma^*(X)\dot{\mathcal{V}}^*(X) = \lambda_0\mathcal{V}^*(X) - u(\sigma^*(X)) + \Delta F(X)\varphi(X) \tag{D.11}$$

$$\sigma^*(X)\dot{\varphi}(X) = \lambda_1(\varphi(X) - \mathcal{V}_{\infty}) + \frac{1}{2}(\dot{\mathcal{V}}^*(X))^2. \tag{D.12}$$

PROOF OF LEMMA D.4: Differentiating (7.3) with respect to X yields

$$\dot{\mathcal{V}}^*(X) = \int_0^{+\infty} e^{-\lambda_0 \tau} \mathcal{Z}^*(\tau; X) u'(\sigma^*(X^*(\tau; X))) \dot{\sigma}^*(X^*(\tau; X)) \frac{\partial X^*}{\partial X}(\tau; X) d\tau$$

$$+ \int_0^{+\infty} e^{-\lambda_0 \tau} \frac{\partial \mathcal{Z}^*}{\partial X}(\tau; X) u(\sigma^*(X^*(\tau; X))) d\tau.$$

Using (D.1), we rewrite this condition as

$$\text{(D.13)} \quad \sigma^*(X)\dot{\mathcal{V}}^*(X) = \int_0^{+\infty} e^{-\lambda_0 \tau} \mathcal{Z}^*(\tau; X) u'(\sigma^*(X^*(\tau; X))) \dot{\sigma}^*(X^*(\tau; X)) \frac{\partial X^*}{\partial \tau}(\tau; X) d\tau$$

$$+ \int_0^{+\infty} e^{-\lambda_0 \tau} \sigma^*(X) \frac{\partial \mathcal{Z}^*}{\partial X}(\tau; X) u(\sigma^*(X^*(\tau; X))) d\tau.$$

Integrating by parts the first integral above, we find

$$\text{(D.14)} \quad \sigma^*(X)\dot{\mathcal{V}}^*(X) = \left[e^{-\lambda_0 \tau} \mathcal{Z}^*(\tau; X) u(\sigma^*(X^*(\tau; X))) \right]_0^{+\infty}$$

$$+\lambda_0 \int_0^{+\infty} e^{-\lambda_0 \tau} \mathcal{Z}^*(\tau; X) u(\sigma^*(X^*(\tau; X))) d\tau$$

$$+ \int_0^{+\infty} e^{-\lambda_0 \tau} \left(\sigma^*(X) \frac{\partial \mathcal{Z}^*}{\partial X}(\tau; X) - \frac{\partial \mathcal{Z}^*}{\partial \tau}(\tau; X) \right) u(\sigma^*(X^*(\tau; X))) d\tau.$$

Notice that

$$\text{(D.15)} \quad \frac{\partial \mathcal{Z}^*}{\partial \tau}(\tau; X) = \Delta(1 - F(X^*(\tau; X)) - \mathcal{Z}^*(\tau; X)).$$

Observe also that

$$\text{(D.16)} \quad \sigma^*(X) \frac{\partial \mathcal{Z}^*}{\partial X}(\tau; X) = -\Delta e^{-\Delta \tau} \int_0^{\tau} f(X^*(s; X)) \frac{\partial X^*}{\partial s}(s; X) e^{\Delta s} ds.$$

Integrating by parts the right-hand side above yields

$$\sigma^*(X) \frac{\partial \mathcal{Z}^*}{\partial X}(\tau; X) = -\Delta e^{-\Delta \tau} \left(F(X^*(\tau; X)) e^{\Delta \tau} - F(X) - \Delta \int_0^{\tau} F(X^*(s; X)) e^{\Delta s} ds \right).$$

Simplifying yields

$$\text{(D.17)} \quad \sigma^*(X) \frac{\partial \mathcal{Z}^*}{\partial X}(\tau; X) = \Delta \left(1 - F(X^*(\tau; X)) - \mathcal{Z}^*(\tau; X)\right) + \Delta e^{-\Delta \tau} F(X).$$

and thus

$$\text{(D.18)} \quad \sigma^*(X) \frac{\partial \mathcal{Z}^*}{\partial X}(\tau; X) - \frac{\partial \mathcal{Z}^*}{\partial \tau}(\tau; X) = \Delta e^{-\Delta \tau} F(X).$$

Inserting (D.18) into (D.14) finally yields (D.11).

Using (7.12), we rewrite (7.11) as

$$\text{(D.19)} \quad \varphi(X) = \mathcal{V}_\infty - \frac{1}{2} \int_0^{+\infty} e^{-\lambda_1 \tau} (\dot{\mathcal{V}}^*(\tilde{X}(\tau; X)))^2 d\tau.$$

Differentiating (D.19) with respect to X yields

$$\dot{\varphi}(X) = - \int_0^{+\infty} e^{-\lambda_1 \tau} \dot{\mathcal{V}}^*(X^*(\tau; X)) \ddot{\mathcal{V}}^*(X^*(\tau; X)) \frac{\partial X^*}{\partial X}(\tau; X) d\tau.$$

Using (D.1), we rewrite this condition as

$$\sigma^*(X)\dot{\varphi}(X) = -\int_0^{+\infty} e^{-\lambda_1\tau}\dot{\mathcal{V}}^*(X^*(\tau;X))\ddot{\mathcal{V}}^*(X^*(\tau;X))\frac{\partial X^*}{\partial\tau}(\tau;X)d\tau. \tag{D.20}$$

Integrating by parts we obtain

$$\int_0^{+\infty} e^{-\lambda_1\tau}\dot{\mathcal{V}}^*(X^*(\tau;X))\ddot{\mathcal{V}}^*(X^*(\tau;X))\frac{\partial X^*}{\partial\tau}(\tau;X)d\tau = \left[\frac{1}{2}e^{-\lambda_1\tau}(\dot{\mathcal{V}}^*(X^*(\tau;X)))^2\right]_0^{+\infty} - \lambda_1(\varphi(X) - \mathcal{V}_\infty).$$

Inserting into (D.20) and taking into account (7.12), we finally obtain (D.12). Finally, taking into account (7.12) and inserting into (D.12) yields (7.13). *Q.E.D.*

Observe that

$$\mathcal{V}(x,\varepsilon;X) = \int_0^{\varepsilon} e^{-\lambda_0\tau}\left(1 - \Delta e^{-\Delta\tau}\int_0^{\tau} F(X+xs)e^{\Delta s}ds\right)u(x)d\tau \tag{D.21}$$

$$+\int_{\varepsilon}^{+\infty} e^{-\lambda_0\tau}\left(1 - \Delta e^{-\Delta\tau}\int_0^{\tau} F(\widetilde{X}(x,\varepsilon,\tau;X))e^{\Delta s}ds\right)u(\sigma^*(\widetilde{X}(x,\varepsilon,\tau;X)))d\tau.$$

From that, we deduce

$$\frac{\partial\mathcal{V}}{\partial\varepsilon}(0,x,X) = u(x) - u(\sigma^*(X)) \tag{D.22}$$

$$+\int_0^{+\infty} e^{-\lambda_0\tau}\mathcal{Z}^*(\tau;X)u'(\sigma^*(X^*(\tau;X)))\dot{\sigma}^*(X^*(\tau;X))\frac{\partial\widetilde{X}}{\partial\varepsilon}(x,\varepsilon,\tau;X)|_{\varepsilon=0}d\tau$$

$$+\int_0^{+\infty} e^{-\lambda_0\tau}\left(-\Delta e^{-\Delta\tau}\int_0^{\tau} f(X^*(s;X))\frac{\partial\widetilde{X}}{\partial\varepsilon}(x,\varepsilon,s;X)|_{\varepsilon=0}e^{\Delta s}ds\right)u(\sigma^*(X^*(\tau;X)))d\tau.$$

Using (D.4), this expression can be simplified as

$$\frac{\partial\mathcal{V}}{\partial\varepsilon}(0,x,X) = u(x) - u(\sigma^*(X)) \tag{D.23}$$

$$+\left(\frac{x}{\sigma^*(X)} - 1\right)\left(\int_0^{+\infty} e^{-\lambda_0\tau}\mathcal{Z}^*(\tau;X)u'(\sigma^*(X^*(\tau;X)))\dot{\sigma}^*(X^*(\tau;X))\frac{\partial X^*}{\partial\tau}(\tau;X)d\tau\right.$$

$$\left.+\int_0^{+\infty} e^{-\lambda_0\tau}\left(-\Delta e^{-\Delta\tau}\int_0^{\tau} f(X^*(s;X))\frac{\partial X^*}{\partial\tau}(s;X)e^{\Delta s}ds\right)u(\sigma^*(X^*(\tau;X)))d\tau\right).$$

Using (D.16) and (D.17), we rewrite

$$\int_0^{+\infty} e^{-\lambda_0\tau}\left(-\Delta e^{-\Delta\tau}\int_0^{\tau} f(X^*(s;X))\frac{\partial X^*}{\partial\tau}(s;X)e^{\Delta s}ds\right)u(\sigma^*(X^*(\tau;X)))d\tau \tag{D.24}$$

$$= \int_0^{+\infty} e^{-\lambda_0\tau}\sigma^*(X)\frac{\partial\mathcal{Z}^*}{\partial X}(\tau;X)u(\sigma^*(X^*(\tau;X)))d\tau.$$

Integrating by parts, we also have

$$\int_0^{+\infty} e^{-\lambda_0\tau}\mathcal{Z}^*(\tau;X)u'(\sigma^*(X^*(\tau;X)))\dot{\sigma}^*(X^*(\tau;X))\frac{\partial X^*}{\partial\tau}(\tau;X)d\tau \tag{D.25}$$

$$= \left[e^{-\lambda_0\tau}\mathcal{Z}^*(\tau;X)u(\sigma^*(X^*(\tau;X)))\right]_0^{+\infty} + \int_0^{+\infty}\left(\lambda_0\mathcal{Z}^*(\tau;X) - \frac{\partial \mathcal{Z}^*}{\partial \tau}(\tau;X)\right)e^{-\lambda_0\tau}u(\sigma^*(X^*(\tau;X)))d\tau.$$

$$= \lambda_0\mathcal{V}^*(X) - u(\sigma^*(X)) - \int_0^{+\infty} e^{-\lambda_0\tau}\frac{\partial \mathcal{Z}^*}{\partial \tau}(\tau;X)u(\sigma^*(X^*(\tau;X)))d\tau.$$

Using (D.24) and (D.25) and inserting into (D.23) yields

$$\frac{\partial \mathcal{V}}{\partial \varepsilon}(0,x,X) = u(x) - u(\sigma^*(X))$$

$$+\left(\frac{x}{\sigma^*(X)} - 1\right)\left(\lambda_0\mathcal{V}^*(X) - u(\sigma^*(X)) - \int_0^{+\infty} e^{-\lambda_0\tau}\left(\sigma^*(X)\frac{\partial \mathcal{Z}^*}{\partial X}(\tau;X) - \frac{\partial \mathcal{Z}^*}{\partial \tau}(\tau;X)\right)u(\sigma^*(X^*(\tau;X)))d\tau\right).$$

Using (D.11), (D.18) and simplifying yields

$$\text{(D.26)} \quad \frac{\partial \mathcal{V}}{\partial \varepsilon}(0,x,X) = u(x) - u(\sigma^*(X)) + (x - \sigma^*(X))\,\dot{\mathcal{V}}^*(X).$$

Because $\frac{\partial \mathcal{V}}{\partial \varepsilon}(0,x,X)$ so obtained is strictly concave in x, the following first-order condition is necessary and sufficient for an interior optimum obtained from (D.8) and (D.9):

$$0 = \frac{\partial^2 \mathcal{V}}{\partial \varepsilon \partial x}(0,\sigma^*(X),X)$$

which writes as (7.12).

Inserting this expression of $\sigma^*(X)$ into (D.11), we now obtain

$$\text{(D.27)} \quad \lambda_0\mathcal{V}^*(X) - \lambda_1\mathcal{V}_\infty + \Delta F(X)\varphi(X) = \zeta\dot{\mathcal{V}}^*(X) + \frac{(\dot{\mathcal{V}}^*(X))^2}{2}.$$

Therefore, $\dot{\mathcal{V}}^*(X)$ is implicitly defined through this second-degree equation. This equation has real solutions provided that the trajectory remains in the domain defined as

$$\text{(D.28)} \quad \lambda_0\mathcal{V}^*(X) - D + \Delta F(X)\varphi(X) \geq 0 \quad \forall X.$$

We will indeed check (see the Proof of Proposition 10 below) that (D.28) always holds along the equilibrium trajectory. Taking then the highest root to (D.27) yields (7.9). As a consequence of (D.33) below, the equilibrium feedback rule (7.12) then ensures that actions always remain within $(0,\zeta) \subset \mathcal{X}$. The lowest root would instead induce negative actions.

To get the limiting behavior (7.10), we prove the following Lemma.

Lemma D.5 *$\mathcal{V}^*(X)$ is non-increasing and satisfies (7.10). $\varphi(X)$ satisfies*

$$\text{(D.29)} \quad \lim_{X\to+\infty}\varphi(X) = \mathcal{V}_\infty.$$

Proof of Lemma D.5: From (D.19), $\varphi(X) \leq \mathcal{V}_\infty$. Thus, we get

$$\dot{\mathcal{V}}^*(X) \leq \zeta\left(-1 + \sqrt{1 + 2\frac{\lambda_0}{\zeta^2}(\mathcal{V}^*(X) - \mathcal{V}_\infty) - 2\frac{\Delta\mathcal{V}_\infty}{\zeta^2}(1 - F(X))}\right).$$

Using the fact that $\sqrt{1+2Y} \leq 1 + Y$ yields

$$\text{(D.30)} \quad \dot{\mathcal{V}}^*(X) \leq -\frac{\Delta\mathcal{V}_\infty}{\zeta}(1 - F(X)) + \frac{\lambda_0}{\zeta}\left(\mathcal{V}^*(X) - \mathcal{V}_\infty\right).$$

Because $\tilde{X}(s;X)\geq X$ and $u(\sigma^*(\tilde{X}(\tau;X)))\leq \lambda_1\mathcal{V}_\infty$, we then obtain

(D.31)
$$\mathcal{V}^*(X)\leq \lambda_1\mathcal{V}_\infty\int_0^{+\infty}e^{-\lambda_0\tau}\left(1-e^{-\Delta\tau}F(X)\left(e^{\Delta\tau}-1\right)\right)d\tau=\mathcal{V}_\infty\left(F(X)+\frac{\lambda_1}{\lambda_0}(1-F(X))\right)\quad\forall X$$

and thus

(D.32) $$\mathcal{V}^*(X)-\mathcal{V}_\infty\leq\frac{\Delta\mathcal{V}_\infty}{\lambda_0}(1-F(X))\quad\forall X.$$

Inserting into (D.30) and simplifying, yields

(D.33) $$\dot{\mathcal{V}}^*(X)\leq 0\quad\forall X.$$

Because $\mathcal{V}^*(X)\geq 0$, $\mathcal{V}^*(X)$ that is non-increasing thus converges when $X\to+\infty$. Let l be this limit. From (D.31), it follows that

(D.34) $$l\leq\mathcal{V}_\infty.$$

Applying Gronwall's Lemma to (D.30) also yields

$$(\mathcal{V}^*(X)-\mathcal{V}_\infty)e^{-\frac{\lambda_0}{\zeta}X}\geq\frac{\Delta\mathcal{V}_\infty}{\zeta}\int_X^{+\infty}(1-F(\tilde{X}))e^{-\frac{\lambda_0}{\zeta}\tilde{X}}d\tilde{X}.$$

Thus

(D.35) $$\mathcal{V}^*(X)\geq\mathcal{V}_\infty+\frac{\Delta\mathcal{V}_\infty}{\zeta}e^{\frac{\lambda_0}{\zeta}X}\int_X^{+\infty}(1-F(\tilde{X}))e^{-\frac{\lambda_0}{\zeta}\tilde{X}}d\tilde{X}.$$

In particular, we have

(D.36) $$\mathcal{V}^*(X)\geq\mathcal{V}_\infty.$$

Taking limits, we also get

(D.37) $$l\geq\mathcal{V}_\infty.$$

Taking together (D.34) and (D.37) yields (7.10). Therefore, we must necessarily have

(D.38) $$\lim_{X\to+\infty}\sigma^*(X)=\zeta.$$

It follows from (7.12) that

(D.39) $$\lim_{X\to+\infty}\dot{\mathcal{V}}^*(X)=0.$$

Inserting (D.38) and (D.39) into (D.11) finally yields (D.29). *Q.E.D.*

Q.E.D.

Proof of Proposition 10: We first observe that with F having finite support $\left[0,\overline{X}\right]$, the solution to the system of first-order differential equations (D.11)-(D.12) together with the feedback rule (7.12) trivially entails

$$\sigma^*(X)=\zeta\quad\forall X\geq\overline{X}$$

and thus

$$\mathcal{V}^*(X) = \varphi(X) = \mathcal{V}_\infty \quad \forall X \geq \overline{X}.$$

The system (7.12)-(D.11)-(D.12) rewrite as (7.9)-(7.13) together with the boundary conditions

$$\text{(D.40)} \quad \mathcal{V}^*(\overline{X}) = \varphi(\overline{X}) = \mathcal{V}_\infty.$$

First, we prove that

$$\text{(D.41)} \quad \varphi(X) \geq 0 \quad \forall X \leq \overline{X}.$$

By definition (7.11), a sufficient condition is to have

$$\text{(D.42)} \quad \frac{1}{2}(\dot{\mathcal{V}}^*(X))^2 \leq \lambda_1 \mathcal{V}_\infty = D + \frac{\zeta^2}{2} \quad \forall X \leq \overline{X}.$$

But from (7.9) and (D.33), we get

$$\text{(D.43)} \quad 0 \leq -\dot{\mathcal{V}}^*(X)) \leq \zeta \quad \forall X \leq \overline{X}$$

and (D.42) holds.

Second, using (D.36) and (D.41), we finally get the lower bound

$$\text{(D.44)} \quad \lambda_0 \mathcal{V}^*(X) - D + \Delta F(X)\varphi(X) \geq \lambda_0 \mathcal{V}_\infty - D \geq 0$$

where the last inequality follows from Footnote 28.

We now transform the system of first-order differential equations (7.9)-(7.13) in *backwards form* by defining $\tilde{\mathcal{V}}^*(Y) = \mathcal{V}^*(X)$, $\tilde{\varphi}(Y) = \varphi(X)$ and $Y = \overline{X} - X \in [0, \overline{X}]$ respectively as

$$\text{(D.45)} \quad \dot{\tilde{\mathcal{V}}}^*(Y) = f_1(Y, \tilde{\mathcal{V}}^*(Y), \tilde{\varphi}(Y)) \equiv \zeta - \sqrt{2\lambda_0 \tilde{\mathcal{V}}^*(Y) - 2D + 2\Delta F(\overline{X} - Y)\tilde{\varphi}(Y)}$$

and

$$\text{(D.46)} \quad \dot{\tilde{\varphi}}(Y) = f_2(Y, \tilde{\mathcal{V}}^*(Y), \tilde{\varphi}(Y)) \equiv -\frac{\lambda_1(\tilde{\varphi}(Y) - \mathcal{V}_\infty) + \frac{1}{2}(\dot{\tilde{\mathcal{V}}}^*(Y))^2}{-\dot{\tilde{\mathcal{V}}}^*(Y) + \zeta}.$$

The boundary conditions (D.40) now become

$$\text{(D.47)} \quad \tilde{\mathcal{V}}^*(0) = \tilde{\varphi}(0) = \mathcal{V}_\infty.$$

By Cauchy-Lipschitz Theorem, there is a unique solution to the system of first-order differential equations (D.45)-(D.46) with the initial conditions (D.47). This theorem only provides existence on an interval of finite length. That this solution can be extended for all $X \in [0, \overline{X}]$ follows from using Wintner Theorem (Wintner, 1946; Nemytskii and Stepanov, 1989 p.11) that provides sufficient conditions for the existence of a global solution valid on $[0, \overline{X}]$ for any arbitrary $\overline{X}$.

Theorem D.1 *Wintner (1946). Suppose that there exists a continuous function $L(r) > 0$ defined for $r \geq r_0$, satisfying*

$$\text{(D.48)} \quad |f_i(Y, \tilde{\mathcal{V}}^*(Y), \tilde{\varphi}(Y))| < L(r), \quad i = 1, 2.$$

and having the property

$$\int_{r_0}^{+\infty} \frac{dr}{L(r)} = +\infty.$$

where $r = \|(\tilde{\mathcal{V}}^*(Y), \tilde{\varphi}(Y))\| \equiv \sqrt{\tilde{\mathcal{V}}^{*2}(Y) + \tilde{\varphi}^2(Y)}$. Then, there is a unique solution to the system of first-order differential equations (D.45)-(D.46) with the initial conditions (D.47) over $[0, \overline{X}]$ for any arbitrary $\overline{X}$.

To apply Wintner Theorem, it is enough to prove that there exist constants c_i and d_i such that the following uniform growth conditions hold:

$$|f_i(Y,\tilde{\mathcal{V}}^*(Y),\tilde{\varphi}(Y)| \leq c_i\|(\tilde{\mathcal{V}}^*(Y),\tilde{\varphi}(Y))\| + d_i, \quad i=1,2. \tag{D.49}$$

Then, taking $L(r) = \max\{c_1,c_2\}r + \max\{d_1,d_2\}$ does the job.

First, we notice that

$$|f_1(Y,\tilde{\mathcal{V}}^*(Y),\tilde{\varphi}(Y))| = |\zeta - \sqrt{\zeta^2 + 2\lambda_0(\tilde{\mathcal{V}}^*(Y) - \mathcal{V}_\infty) + 2\Delta(\tilde{\varphi}(Y) - \mathcal{V}_\infty)}|$$

and thus, using the triangular inequality,

$$|f_1(Y,\tilde{\mathcal{V}}^*(Y),\tilde{\varphi}(Y))| \leq |\zeta| + \left|\sqrt{\zeta^2 + 2\lambda_0(\tilde{\mathcal{V}}^*(Y) - \mathcal{V}_\infty) + 2\Delta(\tilde{\varphi}(Y) - \mathcal{V}_\infty)}\right|.$$

Using now the fact that $\sqrt{1+2Y} \leq 1+Y$, we obtain

$$|f_1(Y,\tilde{\mathcal{V}}^*(Y),\tilde{\varphi}(Y))| \leq \zeta + \zeta\left|1 + \frac{\lambda_0}{\zeta^2}(\tilde{\mathcal{V}}^*(Y) - \mathcal{V}_\infty) + \frac{\Delta}{\zeta^2}(\tilde{\varphi}(Y) - \mathcal{V}_\infty)\right|$$

and thus, using the triangular inequality,

$$|f_1(Y,\tilde{\mathcal{V}}^*(Y),\tilde{\varphi}(Y))| \leq \zeta\left(2 + \frac{\lambda_0}{\zeta^2}(|\tilde{\mathcal{V}}^*(Y)| + \mathcal{V}_\infty) + \frac{\Delta}{\zeta^2}(|\tilde{\varphi}(Y)| + \mathcal{V}_\infty)\right) \tag{D.50}$$

Observing that $|x| + |y| \leq \sqrt{2}\sqrt{|x|^2 + |y|^2}$ (D.50) implies (D.49) for some constants

$$c_1 = \sqrt{2}\max\left\{\frac{\lambda_0}{\zeta}; \frac{\Delta}{\zeta}\right\}; d_1 = \sqrt{2}\left(2\zeta + \frac{\lambda_1}{\zeta}\mathcal{V}_\infty\right).$$

Using (D.36) and taking the backward expression of variables yields

$$\tilde{\mathcal{V}}^*(Y) \geq \mathcal{V}_\infty \quad \forall Y \in \left[0, \overline{X}\right].$$

From this and the fact that $\tilde{\varphi}(Y) \geq 0$ (from the definition (7.11)), observe that

$$-\dot{\tilde{\mathcal{V}}}^*(Y) + \zeta = \sqrt{\zeta^2 + 2\lambda_0(\tilde{\mathcal{V}}^*(Y) - \mathcal{V}_\infty) + 2\Delta(\tilde{\varphi}(Y) - \mathcal{V}_\infty)} \geq \sqrt{2\lambda_0\mathcal{V}_\infty - 2D} > 0 \tag{D.51}$$

where the last inequality follows from (D.44).

Using the triangular inequality, we now have

$$|f_2(Y,\tilde{\mathcal{V}}^*(Y),\tilde{\varphi}(Y))| \leq \frac{\lambda_1}{\sqrt{\zeta^2 + 2\lambda_0(\tilde{\mathcal{V}}^*(Y) - \mathcal{V}_\infty) + 2\Delta(\tilde{\varphi}(Y) - \mathcal{V}_\infty)}}|\tilde{\varphi}(Y) - \mathcal{V}_\infty| + \left|\frac{(\dot{\tilde{\mathcal{V}}}^*(Y) - \zeta + \zeta)^2}{2(-\dot{\tilde{\mathcal{V}}}^*(Y) + \zeta)}\right|.$$

Using (D.51) to provide an upper bound of the first term on the right-hand side yields

$$|f_2(Y,\tilde{\mathcal{V}}^*(Y),\tilde{\varphi}(Y))| \leq \frac{\lambda_1}{\sqrt{2\lambda_0\mathcal{V}_\infty - 2D}}|\tilde{\varphi}(Y) - \mathcal{V}_\infty| + \left|\frac{(\dot{\tilde{\mathcal{V}}}^*(Y) - \zeta + \zeta)^2}{2(-\dot{\tilde{\mathcal{V}}}^*(Y) + \zeta)}\right|.$$

Developing the second term on the right-hand side, we obtain

$$|f_2(Y,\tilde{\mathcal{V}}^*(Y),\tilde{\varphi}(Y))| \leq \frac{\lambda_1}{\sqrt{2\lambda_0\mathcal{V}_\infty - 2D}}|\tilde{\varphi}(Y) - \mathcal{V}_\infty| + \left|\frac{\zeta^2}{2(-\dot{\tilde{\mathcal{V}}}^*(Y) + \zeta)} - \zeta + \frac{-\dot{\tilde{\mathcal{V}}}^*(Y) + \zeta}{2}\right|.$$

Using again the triangular inequality and (D.51) to provide an upper bound of the second term on the right-hand side yields

$$|f_2(Y, \tilde{\mathcal{V}}^*(Y), \tilde{\varphi}(Y))| \leq \frac{\lambda_1}{\sqrt{2\lambda_0\mathcal{V}_\infty - 2D}}|\tilde{\varphi}(Y) - \mathcal{V}_\infty| + \frac{\zeta^2}{2\sqrt{2\lambda_0\mathcal{V}_\infty - 2D}} + \zeta$$

$$+\frac{1}{2}\sqrt{\zeta^2 + 2\lambda_0(\tilde{\mathcal{V}}^*(Y) - \mathcal{V}_\infty) + 2\Delta(\tilde{\varphi}(Y) - \mathcal{V}_\infty)}.$$

Again using the fact that $\sqrt{1+2Y} \leq 1+Y$ and the triangular inequality repeatedly, we now obtain

$$|f_2(Y, \tilde{\mathcal{V}}^*(Y), \tilde{\varphi}(Y))| \leq \frac{\lambda_1}{\sqrt{2\lambda_0\mathcal{V}_\infty - 2D}}(|\tilde{\varphi}(Y)| + \mathcal{V}_\infty) + \frac{\zeta^2}{2\sqrt{2\lambda_0\mathcal{V}_\infty - 2D}} + \frac{3\zeta}{2}$$

$$+\frac{\lambda_0}{2\zeta}(|\tilde{\mathcal{V}}^*(Y)| + \mathcal{V}_\infty) + \frac{\Delta}{2\zeta}(|\tilde{\varphi}(Y)| + \mathcal{V}_\infty)$$

and thus, proceeding as above,

$$\text{(D.52)} \quad |f_2(Y, \tilde{\mathcal{V}}^*(Y), \tilde{\varphi}(Y))| \leq c_2\|\tilde{\mathcal{V}}^*(Y), \tilde{\varphi}(Y))\| + d_2$$

for some constants

$$c_2 = \sqrt{2}\max\left\{\frac{\lambda_0}{2\zeta}; \frac{\lambda_1}{\sqrt{2\lambda_0\mathcal{V}_\infty - 2D}} + \frac{\Delta}{2\zeta}\right\}; d_2 = \sqrt{2}\left(\left(\frac{\lambda_1}{\sqrt{2\lambda_0\mathcal{V}_\infty - 2D}} + \frac{\lambda_1}{2\zeta}\right)\mathcal{V}_\infty + \frac{\zeta^2}{2\sqrt{2\lambda_0\mathcal{V}_\infty - 2D}} + \frac{3\zeta}{2}\right).$$

This ends the proof of global existence of a *backward* solution. By flipping variables, this procedure allows us to reconstruct the *forward* solution $(\mathcal{V}^*(X), \varphi(X))$ over $\left[0, \overline{X}\right]$. *Q.E.D.*

Proof of Proposition 11: We now come back on the system of first-order differential equations (7.9)-(7.13) together with the boundary conditions (7.10)-(D.29). Consider the new variables

$$Y = 1 - F(X) \in [0,1], \mathcal{V}^*(X) - \mathcal{V}_\infty = \mathcal{U}(Y), \varphi(X) - \mathcal{V}_\infty = \psi(Y) \text{ and } R(Y) = f(F^{-1}(1-Y)).$$

We rewrite (7.9)-(7.13) respectively as

$$\text{(D.53)} \quad \dot{\mathcal{U}}(Y) = \frac{1}{R(Y)}\left(\zeta - \sqrt{\zeta^2 + 2\lambda_0\mathcal{U}(Y) - 2\Delta\mathcal{V}_\infty + 2\Delta(1-Y)\mathcal{V}_\infty + 2\Delta(1-Y)\psi(Y)}\right)$$

$$\text{(D.54)} \quad \dot{\psi}(Y) = \frac{-\lambda_1\psi(Y) - \frac{1}{2}(\dot{\mathcal{U}}(Y))^2R^2(Y)}{R(Y)\left(\zeta - \dot{\mathcal{U}}(Y)R(Y)\right)}$$

while the boundary conditions (7.10)-(D.29) become

$$\text{(D.55)} \quad \mathcal{U}(0) = \psi(0) = 0.$$

To analyze the local behavior of this solution in the neighborhood of $Y = 0$, we transform this system as an autonomous system by introducing a new time scale z and express $\mathcal{U}$, φ and Y as functions of z (these functions being now denoted with a *tilda*) so that:

$$\text{(D.56)} \quad \dot{\tilde{\mathcal{U}}}(z) = -\zeta + \sqrt{\zeta^2 + 2\lambda_0\tilde{\mathcal{U}}(z) - 2\Delta\mathcal{V}_\infty\tilde{Y}(z) + 2\Delta(1-\tilde{Y}(z))\tilde{\psi}(z)}$$

(D.57) $\dot{\tilde{\psi}}(z) = \dfrac{\lambda_1 \tilde{\psi}(z) + \frac{1}{2}\dot{\tilde{\mathcal{U}}}^2(z)}{\zeta + \dot{\tilde{\mathcal{U}}}(z)}$

(D.58) $\dot{\tilde{Y}}(z) = -R(\tilde{Y}(z))$

with the boundary conditions

(D.59) $\lim_{z\to+\infty} \tilde{\psi}(z) = \lim_{z\to+\infty} \tilde{\mathcal{U}}(z) = \lim_{z\to+\infty} \tilde{Y}(z) = 0.$

We now linearize this system around its long-run equilibrium $(0,0,0)$ to get:

(D.60) $\dot{\tilde{\mathcal{U}}}(z) = \dfrac{\lambda_0}{\zeta}\tilde{\mathcal{U}}(z) + \dfrac{\Delta}{\zeta}\tilde{\psi}(z) - \dfrac{\Delta\mathcal{V}_\infty}{\zeta}\tilde{Y}(z)$

(D.61) $\dot{\tilde{\psi}}(z) = \dfrac{\lambda_1}{\zeta}\tilde{\psi}(z)$

(D.62) $\dot{\tilde{Y}}(z) = -R'(0)\tilde{Y}(z).$

The properties of the linear system above are thus determined by those of the following matrix

$$A = \begin{pmatrix} \frac{\lambda_0}{\zeta} & \frac{\Delta}{\zeta} & -\frac{\Delta\mathcal{V}_\infty}{\zeta} \\ 0 & \frac{\lambda_1}{\zeta} & 0 \\ 0 & 0 & -R'(0) \end{pmatrix}$$

A has two positive eigenvalues and one negative one. The system is hyperbolic and its equilibrium $(0,0,0)$ is thus a saddle. The plane $(\tilde{\mathcal{U}}, \tilde{\psi})$ is unstable while the axis Y is stable.

From the Hartman-Grobman Theorem (Perko, 1991, Section 2.8), the nonlinear system (D.56)-(D.57)-(D.58) and the linear system (D.60)-(D.61)-(D.62) are topologically equivalent. More formally, let ϕ_z be the flow for the nonlinear system (D.56)-(D.57)-(D.58). Because A has non-zero eigenvalues, there exists a homeomorphism H on an open neighborhood U of $(0,0,0)$, such that for each $(\tilde{\mathcal{U}}_0, \tilde{\psi}_0, Y_0) \in U$ there is an open interval $I_0 \subset \mathbb{R}$ containing zero such that for all $(\tilde{\mathcal{U}}_0, \tilde{\psi}_0, Y_0) \in U$ and $z_0 \in I_0$, $H(\phi_z(\tilde{\mathcal{U}}_0, \tilde{\psi}_0, Y_0)) = e^{Az}H(\tilde{\mathcal{U}}_0, \tilde{\psi}_0, Y_0)$.

From this homeomorphism, it follows that the stable manifold for the nonlinear system (D.56)-(D.57)-(D.58) is also one-dimensional. This means that there is an onto relationship between $\tilde{\mathcal{U}}$ and Y on that manifold. Henceforth, the solution $\mathcal{U}(Y)$ is also unique and thus the value function $\mathcal{V}^*(X)$ is also unique. This ends the proof of uniqueness of the equilibrium.

Approximations.To give an approximation of the solution. Observe that the linear system (D.60)-(D.61)-(D.62) can be solved recursively by noticing first that

(D.63) $\tilde{Y}(z) = \tilde{Y}_0 e^{-R'(0)z}$

for some arbitrary $\tilde{Y}_0$ since all such solutions satisfy (D.59). From (D.61), we also have

(D.64) $\tilde{\psi}(z) = \tilde{\psi}_0 e^{\frac{\lambda_1}{\zeta}z}$

but the only solution consistent with (D.59) has $\tilde{\psi}_0 = 0$. Finally inserting those findings into (D.60), we get

$$\text{(D.65)} \quad \dot{\tilde{\mathcal{U}}}(z) = \frac{\lambda_0}{\zeta}\tilde{\mathcal{U}}(z) - \frac{\Delta\mathcal{V}_\infty \tilde{Y}_0}{\zeta} e^{-R'(0)z}$$

Integrating yields

$$\text{(D.66)} \quad \tilde{\mathcal{U}}(z) = e^{\frac{\lambda_0}{\zeta}z}\left(\tilde{\mathcal{U}}_0 - \frac{\Delta\mathcal{V}_\infty \tilde{Y}_0}{\zeta}\int_0^z e^{-\left(R'(0)+\frac{\lambda_0}{\zeta}\right)\tilde{z}} d\tilde{z}\right).$$

The only solution consistent with (D.59) has thus

$$\text{(D.67)} \quad \tilde{\mathcal{U}}_0 = \frac{\Delta\mathcal{V}_\infty \tilde{Y}_0}{\zeta R'(0) + \lambda_0}$$

which gives us a first-order approximation for the stable manifold. Expressed in terms of our original variable, we find that the one-dimensional stable manifold can be approximated as (7.15) when $X \to +\infty$. *Q.E.D.*

Proof of Proposition 12: Condition (D.35) (resp. (D.32)) immediately yields the left-hand side (resp. the right-hand side) of (7.17). Inserting (D.33) into (7.12) yields the right-hand side of (7.18). Observe now that (7.12) and (7.9) imply

$$\sigma^*(X) = \sqrt{\zeta^2 - 2\Delta\mathcal{V}_\infty + 2\lambda_0(\mathcal{V}^*(X) - \mathcal{V}_\infty) + 2\Delta F(X)\varphi(X)} \geq \sqrt{\zeta^2 - 2\Delta\mathcal{V}_\infty}$$

where the inequality follows from the left-hand side of (7.17). From this, we immediately get the left-hand side of (7.18). *Q.E.D.*

APPENDIX E: THE RELEVANCE OF THE *PRECAUTIONARY PRINCIPLE*

Proof of Proposition 13: Differentiating (8.2) with respect to ε, for $\tau \geq \varepsilon$, yields

$$\frac{\partial \widetilde{X}}{\partial \varepsilon}(x,\varepsilon,\tau;0) = x - \sigma^*(x\varepsilon) + \int_\varepsilon^\tau \dot{\sigma}^*(X^*(s-\varepsilon;x\varepsilon))\left(-\frac{\partial X^*}{\partial s}(s-\varepsilon;x\varepsilon) + x\frac{\partial X^*}{\partial X}(s-\varepsilon;x\varepsilon)\right) ds.$$

or using (D.1)

$$\begin{aligned}\frac{\partial \widetilde{X}}{\partial \varepsilon}(x,\varepsilon,\tau;0) &= x - \sigma^*(x\varepsilon) + \left(\frac{x}{\sigma^*(x\varepsilon)} - 1\right)\int_\varepsilon^\tau \dot{\sigma}^*(X^*(s-\varepsilon;x\varepsilon))\sigma^*(X^*(s-\varepsilon;x\varepsilon)))ds \\ &= x - \sigma^*(x\varepsilon) + \left(\frac{x}{\sigma^*(x\varepsilon)} - 1\right)\int_\varepsilon^\tau \dot{\sigma}^*(X^*(s-\varepsilon;x\varepsilon))\frac{\partial X^*}{\partial s}(s-\varepsilon;x\varepsilon)ds \\ &= x - \sigma^*(x\varepsilon) + \left(\frac{x}{\sigma^*(x\varepsilon)} - 1\right)(\sigma^*(X^*(\tau-\varepsilon;x\varepsilon)) - \sigma^*(x\varepsilon))\end{aligned}$$

and thus

$$\text{(E.1)} \quad \frac{\partial \widetilde{X}}{\partial \varepsilon}(x,\varepsilon,\tau;0) = \begin{cases} 0 & \text{for } \tau < \varepsilon, \\ \left(\frac{x}{\sigma^*(x\varepsilon)} - 1\right)\sigma^*(X^*(\tau-\varepsilon;x\varepsilon)) & \text{for } \tau \geq \varepsilon. \end{cases}$$

From (7.6), we get

(E.2) $$\mathcal{Z}(x,\varepsilon,\tau;0) = 1 - \Delta e^{-\Delta\tau}\int_0^\tau F(\widetilde{X}(x,\varepsilon,s;0))e^{\Delta s}ds.$$

Differentiating (E.2) with respect to ε yields

$$\frac{\partial \mathcal{Z}}{\partial \varepsilon}(x,\varepsilon,\tau;0) = -\Delta e^{-\Delta\tau}\int_0^\tau f(\widetilde{X}(x,\varepsilon,s;0))\frac{\partial \widetilde{X}}{\partial \varepsilon}(x,\varepsilon,s;0)e^{\Delta s}ds$$

and using (E.1) for $\tau \geq \varepsilon$,

(E.3)
$$\frac{\partial \mathcal{Z}}{\partial \varepsilon}(x,\varepsilon,\tau;0) = \begin{cases} 0 & \text{for } \tau < \varepsilon, \\ -\left(\frac{x}{\sigma^*(x\varepsilon)} - 1\right)\Delta e^{-\Delta\tau}\int_0^\tau f(\widetilde{X}(x,\varepsilon,s;0))\sigma^*(X^*(s-\varepsilon;x\varepsilon))e^{\Delta s}ds & \text{for } \tau \geq \varepsilon. \end{cases}$$

Differentiating now (8.1) with respect to ε, we find

$$\frac{\partial \mathcal{V}}{\partial \varepsilon}(x,\varepsilon;0) = e^{-\lambda_0\epsilon}\mathcal{Z}(\varepsilon,\varepsilon;0)(u(x) - u(\sigma^*(x\varepsilon)))$$
$$+\int_\varepsilon^{+\infty} e^{-\lambda_0\tau}\mathcal{Z}(x,\varepsilon,\tau;0)u'(\sigma^*(\widetilde{X}(\varepsilon,\tau;x\varepsilon)))\dot{\sigma}^*(\widetilde{X}(\varepsilon,\tau;x\varepsilon))\frac{\partial \widetilde{X}}{\partial \varepsilon}(x,\varepsilon,\tau;0)d\tau$$
$$+\int_\varepsilon^{+\infty} e^{-\lambda_0\tau}\frac{\partial \mathcal{Z}}{\partial \varepsilon}(x,\varepsilon,\tau;0)u(\sigma^*(\widetilde{X}(\varepsilon,\tau;x\varepsilon)))d\tau.$$

Using (E.1) and (E.3) into the first and second integrals above immediately yields

$$\frac{\partial \mathcal{V}}{\partial \varepsilon}(\varepsilon,x,0) = 0$$

when (8.3) holds.

Let denote the solution $x^*(\varepsilon)$ so implicitly defined. Because $\sigma^*(0) > 0$ and $\sigma^*(X) < \zeta$ for all X, there exists always such a solution $x^*(\varepsilon) > 0$ for any $\varepsilon > 0$. Moreover, the Implicit Function Theorem implies $\dot{x}^*(\varepsilon) = \frac{x^*(\varepsilon)\dot{\sigma}^*(x^*(\varepsilon))}{1-\varepsilon\dot{\sigma}^*(x^*(\varepsilon))}$. In particular, we get

(E.4) $$x(0) = \sigma^*(0) \text{ and } \dot{x}(0) = \sigma^*(0)\dot{\sigma}^*(0).$$

Differentiating (8.2) with respect to x, for $\tau \geq \varepsilon$, yields

$$\frac{\partial \widetilde{X}}{\partial x}(x,\varepsilon,\tau;0) = \varepsilon\left(1 + \int_\varepsilon^\tau \dot{\sigma}^*(X^*(s-\varepsilon;x\varepsilon))\frac{\partial X^*}{\partial X}(s-\varepsilon;x\varepsilon)ds\right).$$

or using (D.1)

$$\frac{\partial \widetilde{X}}{\partial x}(x,\varepsilon,\tau;0) = \varepsilon\left(1 + \frac{1}{\sigma^*(x\varepsilon)}\int_\varepsilon^\tau \dot{\sigma}^*(X^*(s-\varepsilon;x\varepsilon))\sigma^*(X^*(s-\varepsilon;x\varepsilon)))ds\right)$$
$$= \varepsilon\left(1 + \frac{1}{\sigma^*(x\varepsilon)}\int_\varepsilon^\tau \dot{\sigma}^*(X^*(s-\varepsilon;x\varepsilon))\frac{\partial X^*}{\partial s}(s-\varepsilon;x\varepsilon)ds\right)$$

and thus

(E.5) $$\frac{\partial \widetilde{X}}{\partial x}(x,\varepsilon,\tau;0) = \begin{cases} \tau & \text{for } \tau \leq \varepsilon, \\ \varepsilon\frac{\sigma^*(X^*(\tau-\varepsilon;x\varepsilon))}{\sigma^*(x\varepsilon)} & \text{for } \tau \geq \varepsilon. \end{cases}$$

Observe in particular that we get the following first-order Taylor expansion in ε in the neighborhood of $\varepsilon = 0$,

$$\frac{\partial \widetilde{X}}{\partial x}(x, \varepsilon, \tau; 0) = \varepsilon \frac{\sigma^*(X^*(\tau; 0))}{\sigma^*(0)}. \tag{E.6}$$

Differentiating (E.2) with respect to x yields

$$\frac{\partial \mathcal{Z}}{\partial x}(x, \varepsilon, \tau; 0) = -\Delta e^{-\Delta\tau} \int_0^\tau f(\widetilde{X}(x, \varepsilon, s; 0)) \frac{\partial \widetilde{X}}{\partial x}(x, \varepsilon, s; 0) e^{\Delta s} ds$$

and using (E.5)

$$\frac{\partial \mathcal{Z}}{\partial x}(x, \varepsilon, \tau; 0) = \begin{cases} -\Delta e^{-\Delta\tau} \int_0^\tau f(sx) s e^{\Delta s} ds & \text{for } \tau \leq \varepsilon, \\ -\frac{\varepsilon}{\sigma^*(x\varepsilon)} \Delta e^{-\Delta\tau} \int_0^\tau f(\widetilde{X}(x, \varepsilon, s; 0)) \sigma^*(X^*(s - \varepsilon; x\varepsilon)) e^{\Delta s} ds & \text{for } \tau \geq \varepsilon. \end{cases} \tag{E.7}$$

From that, we get the following first-order Taylor expansion in ε in the neighborhood of $\varepsilon = 0$,

$$\frac{\partial \mathcal{Z}}{\partial x}(x, \varepsilon, \tau; 0) = -\frac{\varepsilon}{\sigma^*(0)} \Delta e^{-\Delta\tau} \int_0^\tau f(X^*(s; 0)) \sigma^*(X^*(s; 0)) e^{\Delta s} ds. \tag{E.8}$$

Now observe that

$$\frac{\partial \mathcal{Z}^*}{\partial \tau}(\tau; 0) = -\Delta F(X^*(\tau; 0)) + \Delta^2 e^{-\Delta\tau} \int_0^\tau F(X^*(s; 0)) e^{\Delta s} ds. \tag{E.9}$$

Integrating by parts, we also get

$$-\Delta e^{-\Delta\tau} \int_0^\tau f(X^*(s; 0)) \frac{\partial X^*}{\partial \tau}(s; 0) e^{\Delta s} ds = -\Delta F(X^*(\tau; 0)) + \Delta^2 e^{-\Delta\tau} \int_0^\tau F(X^*(s; 0)) e^{\Delta s} ds. \tag{E.10}$$

Inserting into (E.8) and (E.9) finally yields the following first-order Taylor expansion in ε in the neighborhood of $\varepsilon = 0$,

$$\frac{\partial \mathcal{Z}}{\partial x}(x, \varepsilon, \tau; 0) = \frac{\varepsilon}{\sigma^*(0)} \frac{\partial \mathcal{Z}^*}{\partial \tau}(\tau; 0). \tag{E.11}$$

Differentiating now (8.1) with respect to x, we find

$$\frac{\partial \mathcal{V}}{\partial x}(x, \varepsilon; 0) = \int_0^{+\infty} e^{-\lambda_0\tau} \mathcal{Z}(x, \varepsilon, \tau; 0) u'(\sigma^*(\widetilde{X}(\varepsilon, \tau; x\varepsilon))) \dot{\sigma}^*(\widetilde{X}(\varepsilon, \tau; x\varepsilon)) \frac{\partial \widetilde{X}}{\partial x}(x, \varepsilon, \tau; 0) d\tau \tag{E.12}$$

$$+ \int_0^{+\infty} e^{-\lambda_0\tau} \frac{\partial \mathcal{Z}}{\partial x}(x, \varepsilon, \tau; 0) u(\sigma^*(\widetilde{X}(\varepsilon, \tau; x\varepsilon))) d\tau.$$

Let us now denote $\omega(\varepsilon) = \mathcal{V}(x^*(\varepsilon), \varepsilon; 0)$. By the Envelope Theorem, we get

$$\dot{\omega}(\varepsilon) = \frac{\partial \mathcal{V}}{\partial x}(x^*(\varepsilon), \varepsilon; 0) \dot{x}^*(\varepsilon).$$

Using (E.6), (E.11) and (E.12), we compute the following first-order Taylor expansion

$$\frac{\partial \mathcal{V}}{\partial x}(\sigma^*(0), \varepsilon; 0) = \varepsilon \frac{\dot{x}(0)}{\sigma^*(0)} \Big(\int_0^{+\infty} e^{-\lambda_0\tau} \mathcal{Z}^*(\tau; 0) u'(\sigma^*(X^*(\tau; 0)) \dot{\sigma}^*(X^*(\tau; 0))) \frac{\partial X^*}{\partial \tau}(\tau; 0) d\tau \tag{E.13}$$

$$+\int_0^{+\infty} e^{-\lambda_0\tau}\frac{\partial \mathcal{Z}^*}{\partial \tau}(\tau;0)u(\sigma^*(X^*(\tau;0)))d\tau\Big).$$

Integrating by parts, we get

$$\text{(E.14)}\quad \int_0^{+\infty} e^{-\lambda_0\tau}\mathcal{Z}^*(\tau;0)u'(\sigma^*(X^*(\tau;0))\dot{\sigma}^*(X^*(\tau;0))\frac{\partial X^*}{\partial \tau}(\tau;0)d\tau = -u(\sigma^*(0))$$

$$-\int_0^{+\infty} e^{-\lambda_0\tau}\left(\frac{\partial \mathcal{Z}^*}{\partial \tau}(\tau;0) - \lambda_0\mathcal{Z}^*(\tau;0)\right)u(\sigma^*(X^*(\tau;0))d\tau.$$

Inserting within (E.13) and simplifying yields the following first-order Taylor expansion in ε

$$\text{(E.15)}\quad \frac{\partial \mathcal{V}}{\partial x}(\sigma^*(0),\varepsilon;0) = \varepsilon\frac{\dot{x}(0)}{\sigma^*(0)}\Big(-u(\sigma^*(0)) + \lambda_0\mathcal{V}^*(0)\Big).$$

Using (D.11) and (E.4) yields the following first-order Taylor expansion in ε

$$\frac{\partial \mathcal{V}}{\partial x}(\sigma^*(0),\varepsilon;0) = \varepsilon\sigma^*(0)\dot{\sigma}^*(0)\dot{\mathcal{V}}^*(0).$$

Because $\dot{\mathcal{V}}^*(0) < 0$, $\frac{\partial \mathcal{V}}{\partial x}(\sigma^*(0),\varepsilon;0) > 0$ in a right-neighborhood of $\varepsilon = 0$ if and only if $\dot{\sigma}^*(0) < 0$. *Q.E.D.*