Monetary Policy, Capital Controls, and International Portfolios*

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November 10, 2023

Abstract

I study optimal monetary policy and capital controls in a small open economy model with nominal rigidities, incomplete markets, and cross-border holdings of assets denominated in home and foreign currency. Monetary policy can enhance risk sharing across countries by influencing exchange rates. The strength of this channel depends on the international portfolio, giving rise to a potential rationale for capital controls. I develop an approximation method that allows me to characterize the optimal policy explicitly. I show that optimal monetary policy is a weighted average of an inflation target and an insurance target and characterize the optimal weight sharply. Perhaps surprisingly, as insurance considerations become more important, home-currency positions become larger, and the realized excess return volatility of home-currency assets actually decreases, rather than increases as one would expect with exogenous portfolios. In addition, I find that private portfolio decisions in small open economies are approximately efficient so that differential capital controls on foreign- vs. home-currency assets are not necessary.

JEL codes: E52, F31, F41

Keywords: Optimal Monetary Policy, Exchange Rate Volatility, Incomplete Markets, Portfolio Choice, Capital Controls, Macroprudential Policy

1 Introduction

The size of international balance sheets has increased dramatically in the past three decades (Lane and Milesi-Ferretti, 2007). In this context, small movements in exchange rates, stock, and bond prices can create large capital gains and losses across borders. Today, these *valuation effects* are often of comparable magnitude to current account fluctuations (Gourinchas and Rey, 2013; Lane

*I am very grateful to my advisors Iván Werning, Arnaud Costinot, and Ricardo Caballero for their continuous guidance and support. I thank Florencia Airaudo, Dmitry Mukhin and Emmanuel de Veirman for their discussions, and Mark Aguiar, Manuel Amador, Marios Angeletos, Anmol Bhandari, Vivek Bhattacharya, Javier Bianchi, Giancarlo Corsetti, José Fanelli, Daniel Greenwald, Jonathan Heathcote, Greg Howard, Oleg Itskhoki, Anastasios Karantounias, Margarita Machelett, Dmitry Mukhin, Diego Perez, Fabrizio Perri, Alp Simsek, Ludwig Straub, Adrien Verdelhan, and Olivier Wang for helpful comments. I also thank the participants at several academic seminars and conferences. This work is part of the grant RYC2021-034948-I, financed by MCIN/AEI/10.13039/501100011033 and the European Union «NextGenerationEU»/PRTRI. *First version:* February 2017. All errors are my own. *Email:* fanelli@cemfi.es.

and Milesi-Ferretti, 2007; Tille and van Wincoop, 2010). As a result, they are crucial to understand the behavior of a country's international invesment position. However, the policy implications of this phenomenon remain largely unexplored.

In this paper, I study the implications of financial integration for optimal monetary policy and capital controls. The focus on these two policy instruments is motivated by two observations. The first observation is that both monetary policy and capital controls can be used to influence exchange rate movements, which are one of the most important sources of asset-price fluctuations in open economies (Lane and Shambaugh, 2010). For example, tightening monetary policy and taxing savings typically leads to a stronger currency, increasing the return of home-currency bonds. Thus, by increasing the returns of the country's international portfolio in bad times, and decreasing them in good times, central bank policies can improve the hedging properties of the portfolio; that is, they can play an *insurance* role.[1]

The second observation is that a country's international portfolio is the key determinant of the strength of the insurance channel. When agents have sizeable cross-border positions in home-currency assets, exchange rate movements can be very powerful as a means of completing markets. This has two implications. First, there is a two-way feedback between monetary policy and portfolio choice, as positions depend on agents' expectations of monetary policy. Second, capital controls taxing the composition of international portfolios may be desirable, as agents do not internalize the effect of their portfolio choice on the ability of the central bank to provide insurance. Indeed, the presence of incomplete markets and nominal rigidities guarantees this will be the case (Geanokoplos and Polemarchakis, 1986, Farhi and Werning, 2016). However, there is little guidance as to how important these taxes may be or even what sign they may have.

The main contribution of this paper is to characterize optimal monetary policy and capital controls in a model that allows for the previous considerations. From an economic standpoint, this requires: (i) extending the typical open economy macroeconomic model used for optimal policy analysis, where either markets are complete and there is no insurance role, or there is a single asset and there is no role for portfolio choice; and (ii) developing new tools to study optimal policy in these richer environments, where the standard linear-quadratic framework cannot immediately be applied due to the indeterminacy of the portfolio at the steady state. To this end, I extend a canonical open economy model by allowing the home country to trade multiple assets with the rest of the world. I assume that these assets are insufficient to span the whole state space (i.e. markets are incomplete) and that the return of some of these assets depends on monetary policy. I overcome the indeterminacy of the steady-state portfolio by showing how the perturbation approach in Judd and Guu (2001) employed in positive analysis can be used to extend the linear-quadratic normative framework in Benigno and Woodford (2012).

The main results in this paper arise from the interaction between exchange rate management and international portfolio choice. To illustrate the forces at play in the simplest possible way, I

[1]It is well understood that monetary policy can play an insurance role in environments with incomplete markets by affecting the terms of trade (Obstfeld and Rogoff, 2002, Corsetti, Dedola and Leduc, 2010). I abstract from this channel by focusing on a small open economy that faces exogenous terms of trade.

start with a static small open economy model where agents have an endowment of tradable goods and produce nontradable goods with labor. There are two periods.[2] In the first period, agents only trade financial assets. In the second period, the state of the world is realized, agents produce, honor their financial obligations, and consume. The model has two key ingredients. First, like in the canonical model, there are nominal rigidities (sticky prices). This ingredient gives rise to the traditional demand-management role for monetary policy, which can undo the distortions associated with sticky prices. The second ingredient is the availability of home- and foreign-currency bonds that can be traded internationally. This ingredient gives rise to the insurance channel discussed above, and a nontrivial portfolio problem.

In this environment, I study the problem of a planner that maximizes the utility of home households under commitment. The planner has two tools: monetary policy and capital controls. Monetary policy is a state-contingent exchange rate rule. Capital controls are taxes on financial assets. My approximation method allows me to get closed form solutions for the optimal monetary policy, portfolio, and capital controls around the nonstochastic steady state. Using this approximation, the two ingredients described above translate into two targets for monetary policy: a demand-management target and an insurance target. The former is the exchange rate that would be required to attain a zero output gap (i.e. restores production efficiency). The latter is the one that equalizes the marginal utility of tradables at home and abroad (i.e. replicates complete markets).

The optimal monetary policy balances these two objectives. I show that the optimal weight on the insurance target increases with the size of gross positions. The reason is that large gross positions make the return of the portfolio sensitive to monetary policy. That is, small exchange rate movements create substantial capital flows. As a result, the planner can enhance risk sharing at a small cost in terms of the output gap. Conversely, letting the exchange rate float freely to close the output gap is very costly, as this would imply large undesirable transfers of wealth across borders. In other words, currency mismatches endogenously create "fear of floating" in some states of the world, e.g. after non-tradable productivity shocks.

I show three main results that emerge from studying endogenous portfolio choice. The first result is that the planner chooses the portfolio to align the two targets as much as possible but, forced to prioritize an objective, the planner chooses larger gross positions when the insurance motive is more important. This makes it ex post more difficult to stabilize demand, but improves the ability of the planner to provide insurance.

When is the insurance objective more important? There are two main determinants. First, there are primitives that control whether one objective dominates over the other, e.g. the degree of risk aversion or price stickiness. Second, there is the relative likelihood of different kinds of shocks. For example, in the simple model shocks to nontradable productivity require that the exchange rate moves to close the output gap (the demand-management target) but that it stays constant to prevent transfers of wealth (the insurance target). Shocks to the tradable endowment create

[2] I refer to this as the static model because there is no intertemporal decision.

the opposite pattern. Thus, the more volatile the tradable endowment, the more important the insurance motive is. In other words, although risk is a second-order phenomenon, it matters for optimal monetary policy to first order via the portfolio decision.

The second result is that endogenous portfolio choice is crucial for the volatility of home-currency returns. That is, suppose that the planner now cares more about insurance, e.g. prices become more flexible, what happens to the volatility of the home-currency returns? Holding the portfolio constant, the optimal policy prescribes that home-currency returns move more in states of the world where there is a demand for insurance, e.g. after tradable-endowment shocks, but move less otherwise, e.g. after non-tradable productivity shocks. Crucially, I show that this composition effect is exactly zero when evaluated at the optimal portfolio. Since gross positions increase under the optimal policy, the volatility of home-currency returns relative to a pure demand-management policy decreases: a smaller change is need to provide insurance and it becomes more costly to close the output gap.[3]

The third main result is that the approximate solution implies that capital controls should not tax the composition of international flows, despite the presence of aggregate-demand externalities. While private agents would choose the wrong portfolio absent taxes (Farhi and Werning, 2016), as risk vanishes the portfolio that they would choose converges to the socially-optimal portfolio. The key observation behind this result is that eliminating production inefficiencies in this economy is feasible, i.e. the planner can always close the output gap. As a result, the economy only experiences booms and recessions because the planner is trying to improve international risk sharing. Formally, this implies that output gaps are proportional to social marginal utility. Furthermore, the wedge between social and private marginal utility is also proportional to the output gap. These observations imply that social and private marginal utilities are proportional to one another, which is enough to establish the asymptotic optimality of the private portfolio decision.

After solving the simple model, I consider a general framework that allows for a large class of preferences, technology, multiple assets, and a potentially finite number of foreign arbitrageurs. Three main lessons emerge. First, the analysis carries over in terms of two sufficient statistics: the transfers that would arise under complete markets, which capture the potential insurance benefits, and the realized returns in an economy where foreigners do not hold home-currency bonds, which capture the potential fear-of-floating costs associated with currency mismatches. Second, when there are multiple assets, one needs to correct the aforementioned sufficient statistics and subtract the insurance the planner can get "for free", i.e. using the assets at their disposal without distorting production efficiency. Third, limited participation in home-currency bond markets creates a demand for cooperation in monetary and capital-control policy across borders. A selfish home planner would manipulate asset prices to favor the home country, which involves putting non-zero approximate

[3]Note that this result is described with respect to the volatility of the home-currency returns under demand-management. However, the volatility of the returns under a pure demand-management policy itself also changes with the portfolio. Interestingly, this effect depends on the sign of the position: it decreases volatility when home agents are short the home-currency bond and vice versa. Thus, in the empirically-relevant case where the home country is short the home-currency against the rest of the world (Bénétrix et al., 2019), both effects go in the same direction and overall volatility decreases.

taxes to lower international risk sharing.[4]

Next, I study a dynamic version of this economy. I show analytically the robustness of the results and derive new insights, which stem from the fact that the planner has more than one way to affect the return of the country's portfolio. For example, to increase the return of the home-currency bond, the planner can either appreciate the exchange rate today or promise to do so in the future (or both). In addition, the planner can achieve this appreciation using monetary policy and savings taxes. Generically, the planner relies on both policy instruments. In other words, while the composition of capital flows should not be taxed, net flows should, i.e. taxes should be uniform across assets in the approximate solution. The optimal policy mix depends on features of the environment, such as the maturity of the home-currency bond.

Finally, I calibrate the model to explore numerically the quantitative relevance of the results. Since the theoretical analysis assumes commitment, I take Canada as a benchmark advanced small open economy. The numerical results illustrate that modelling endogenous portfolio choice is crucial. Indeed, a key feature of the optimal policy is that the home economy takes more debt in its own currency against the rest of the world. Without this lever, i.e. if the portfolio were exogenously fixed at the calibrated value, the weight on the insurance target under the optimal policy would be about seven times smaller and the welfare gains of financial integration almost halved.

Related literature This paper belongs to a large literature exploring deviations of optimal monetary policy from inflation targeting in New Keynesian open economy models, surveyed by Corsetti, Dedola and Leduc (2010). In particular, my analysis is closely related to papers where monetary policy plays an insurance role linked to the composition of international portfolios. Benigno (2009*a*) and Benigno (2009*b*) characterize the optimal policy in an economy with home- and foreign-currency bonds, but lack endogenous portfolio choice. Chang and Velasco (2006) and Senay and Sutherland (2019) compare the performance of a set of policy rules with optimal portfolio choice. Devereux and Sutherland (2008) study optimal monetary policy, but in a special case where there are enough assets to replicate the first-best allocation. I contribute to this literature by solving the joint optimal monetary and capital-control policy in a second-best environment.

Second, this paper contributes to the literature that studies environments where the planner can control both the country's portfolio (directly, i.e. government debt, or indirectly, via regulation) and monetary policy. Closest to my work is Farhi and Werning (2016), who study a static small-open economy with home- and foreign-currency debt in one of their applications. They provide a formula for portfolio taxes, pointing out that they are generically nonzero. Perhaps surprisingly, I show that, as risk vanishes, these taxes converge to zero faster than the risk premium, so no taxes are needed in the approximate solution. Du, Pflueger and Schreger (2020), Engel and Park (2022), and Ottonello and Perez (2019) study the interaction of monetary policy and the currency denomination of sovereign debt. Drenik, Kirpalani and Perez (2022) study the interaction between the currency denomination of contracts by private agents and monetary policy, chosen by a government that lacks

[4]This result is related to Costinot, Lorenzoni and Werning (2014), except that consumption is optimally procyclical across states (i.e., imperfect risk sharing) instead of over time - see section 4.

commitment. This paper complements these studies by analyzing the problem under commitment.

Third, there is a related closed-economy literature that studies the potential of monetary policy to complete markets with nominal assets in environments with commitment; see Schmitt-Grohe and Uribe (2004), Siu (2006), Lustig, Sleet and Yeltekin (2008) and Sheedy (2014). In these papers, a similar trade-off between demand-management and insurance emerges, but insurance takes place between the government and the private sector, or between borrowers and savers. In addition, my analysis emphasizes the role of exchange rate movements, and the portfolio decision between home- and foreign-currency bonds, which is absent from these studies.

Finally, this paper makes a methodological contribution to the literature on optimal portfolio choice in dynamic stochastic general equilibrium models (Devereux and Sutherland, 2011; Evans and Hnatkovska, 2012; Tille and van Wincoop, 2010). These papers are positive, aiming to approximate the competitive equilibrium given a policy rule. I extend these methods to tackle normative questions. That is, I show how the abstract linear-quadratic optimal policy framework of Benigno and Woodford (2012) can be adapted to handle problems with portfolio choice.

Layout The paper is organized as follows. Section 2 presents a two-period model of a small open economy that trades home- and foreign-currency assets with the rest of the world. I derive the planning problem and describe the approximation method. Section 3 characterizes the optimal policy in this environment. Section 4 extends the two-period model along several dimensions, allowing for large economies and general preferences, technology, and financial assets. Section 5 studies a dynamic version of the model. Section 6 numerically explores the quantitative importance of the results in a calibrated model. Section 7 concludes.

Appendix A contains a general proof for the approximation of planning problems with portfolio choice used in this paper, extending the abstract framework of Benigno and Woodford (2012), which may be of independent interest. Appendix B contains all other proofs and additional extensions. Appendix C contains additional numerical exercises with the calibrated model.

2 Static model

I start with a simple two-period model to illustrate the main analytical results. Since there are no savings decisions, I henceforth refer to this model as the static model. I generalize these results along several dimensions in sections 4 and 5 and appendixes B.4 and B.5.

2.1 Setup

At $t = 0$, agents trade financial assets. At $t = 1$, a state of the world $s \in S$ is realized, agents honor their financial obligations, produce, and consume.[5]

[5] My approximation in section 2.3 puts restrictions on S. I parametrize shocks as $\xi_s = \epsilon u_s$ where $\epsilon > 0$ is a scalar with the interpretation of risk and take the limit $\epsilon \to 0$. The implicit function theorem and the bifurcation theorem I rely on require that u_s is bounded-vector stochastic process (see appendix A).

Home households There is a continuum of households in the home country, maximizing a Greenwood–Hercowitz–Huffman (GHH) utility function

$$\sum_s \pi_s U(C_{Ts}, C_{Ns}, L_s) = \sum_s \pi_s \ln\left(\kappa C_{Ts}^{\alpha} C_{Ns}^{1-\alpha} - \frac{1-\alpha}{1+\varphi} L_s^{1+\varphi}\right), \tag{1}$$

where $\alpha > 0$, π_s is the probability of state s, C_{Ts} is tradable consumption, C_{Ns} is nontradable consumption, L_s is labor, and $\kappa \equiv \alpha^{-\alpha}(1-\alpha)^{-(1-\alpha)}$ is a normalization constant.

At $t = 0$, agents can trade two assets with foreigners: home-currency bonds B, which offer a fixed payment R in home currency, and bonds B^*, which offer a fixed payment of 1 in foreign currency. The budget constraint is given by

$$(1 + \tau_B)B + B^* = T_0,$$

where τ_B is an ad valorem tax on home bonds and T_0 is a lump-sum transfer from the central bank. I assume positions are bounded by $\bar{K} > 0$, i.e. $|B| \leq \bar{K}$.[6]

At $t = 1$, the state of the world s is realized and agents receive a tradable endowment Y_{Ts}. I normalize the foreign-currency price of the tradable good P_{Ts}^* to 1. The budget constraint in state s is

$$C_{Ts} + E_s^{-1} P_{Ns} C_{Ns} = Y_{Ts} + E_s^{-1} W_s L_s + E_s^{-1} \Pi_{Ns} + R E_s^{-1} B + B^*$$

where P_{Ns} is the price of nontradables, W_s is the wage, and Π_{Ns} are profits from nontradable good producers, and E_s is the nominal exchange rate in units of home currency per unit of foreign currency. Optimization over tradable and nontradable consumption and labor yields

$$(\frac{\alpha}{1-\alpha})\frac{C_{Ns}}{C_{Ts}} = \frac{E_s}{P_{Ns}}, \tag{2}$$

$$\kappa^{-1} \frac{C_{Ns}^{\alpha}}{C_{Ts}^{\alpha}} L_s^{\varphi} = \frac{W_s}{P_{Ns}}. \tag{3}$$

Asset optimization yields a no-arbitrage condition,

$$\sum_s \pi_s \left[\left((1+\tau_B)^{-1} R E_s^{-1} - 1\right) \frac{\partial U}{\partial C_T}(s)\right] = 0. \tag{4}$$

Nominal rigidities A representative firm produces nontradable goods with labor using a linear technology,

$$Y_{Ns} = Z_s L_s,$$

[6] I normalize the exchange rate at $t = 0$, $E_0 = 1$, the return of the foreign bond, $R^* = 1$, and the tax on foreign-currency debt, $\tau_{B^*} = 0$. This is without loss of generality in the static model because there is no consumption or production at $t = 0$; i.e. only the relative price of the assets is determined in equilibrium.

where $Z_s > 0$ is nontradable productivity, which may vary across states. The price of the nontradable good is fixed at one

$$P_{Ns} = 1 \ \forall s \in S. \tag{5}$$

Firms satisfy any demand at this price.

Foreign households The home economy is assumed to be small relative to the rest of the world, i.e. its actions do not affect foreign consumption C_s^*. Foreigner optimization implies the following no-arbitrage condition,[7]

$$\sum_s \pi_s \left[(RE_s^{-1} - 1) \frac{dU^*}{dC^*}(s) \right] = 0. \tag{6}$$

Central bank The central bank in the economy has two tools: monetary policy and capital controls. Monetary policy is a state-contingent exchange rate policy rule $\{E_s\}_s$.[8] Capital controls in this model are represented by the portfolio tax τ_B. The proceeds are then rebated to home households through lump-sum transfers,

$$T_0 = \tau_B B. \tag{7}$$

The monetary authority announces the monetary and tax policies at the beginning of time, *before* agents engage in bond trading, and is assumed to be perfectly credible.

Goods and labor market clearing Replacing profits, labor income and the $t = 0$ budget constraint into the $t = 1$ budget constraint, I obtain

$$C_{Ts} = Y_{Ts} + (RE_s^{-1} - 1)B. \tag{8}$$

The market clearing condition for nontradables is given by

$$C_{Ns} = Z_s L_s. \tag{9}$$

Next, I formally define a competitive equilibrium in this economy.

Definition 1. Given a Central Bank policy ($\{E_s\}_s$,τ_B,T_0), an allocation ($\{C_{Ts}\}_s$,$\{C_{Ns}\}_s$,$\{L_s\}_s$,B) together with prices ($\{P_{Ns}\}_s$,$\{W_s\}_s$,R) is a ***competitive equilibrium*** if and only if they solve (2)−(9).

[7] I assume it is infeasible to have a *state-contingent* tax on the *returns* of financial assets by *foreigners*. Otherwise, the planner would attain the first best: the planner completes markets by choosing the tax such that the home return after taxes is equal to the desired transfer of wealth under complete markets.

[8] As is standard in the New Keynesian literature, I focus on the cashless limit (see e.g. Woodford, 2003).

2.2 Planning problem

The planner in the economy is the central bank, which chooses a state-contingent exchange rate $\{E_s\}$ and capital controls τ_B to maximize the utility of home households. Combining equations (2), (5) and (9),

$$C_{Ns} = \frac{1-\alpha}{\alpha} E_s C_{Ts}, \tag{10}$$

$$L_s = \frac{1-\alpha}{\alpha} Z_s^{-1} E_s C_{Ts}. \tag{11}$$

These equations show the allocations of labor and nontradable production that the planner can attain with a policy $\{E_s\}_s$ given $\{C_{Ts}\}_s$. Furthermore, τ_B allows the planner to control the agents' portfolio decision, so (4) can be dropped from the planning problem. Thus, the foreign no-arbitrage condition (6) and the budget constraint (8) characterize the set of implementable allocations.

Lemma 1. *An allocation for tradable consumption $\{C_{Ts}\}_s$, an exchange-rate policy $\{E_s\}_s$, a portfolio B and a home-currency yield R form part of an equilibrium if and only if they solve (6) and (8).*

Next, use equations (10) and (11) to substitute out C_{Ns} and L_s in home's utility function,

$$V(C_{Ts}, E_s; Z_s) = \ln\left(\alpha^{-1} C_{Ts} E_s^{1-\alpha} - \frac{1-\alpha}{1+\varphi}\left(\frac{1-\alpha}{\alpha} Z_s^{-1} E_s C_{Ts}\right)^{1+\varphi}\right) \tag{12}$$

Problem 1. The planner's problem is choosing $\{C_{Ts}\}_s$, $\{E_s\}_s$, and B to maximize

$$\mathcal{W} = \sum_s \pi_s V(C_{Ts}, E_s; Z_s)$$

subject to

$$Y_{Ts} + (RE_s^{-1} - 1)B = C_{Ts}$$

$$\sum_s \pi_s \left[(RE_s^{-1} - 1)\frac{dU^*}{dC^*}(s)\right] = 0.$$

Before tackling this problem, it is useful to study the problem with complete markets.

Problem 2. When markets are complete, the planner's problem is choosing $\{\mathcal{T}_s\}_s$, and $\{E_s\}_s$ to maximize

$$\sum_s \pi_s V(Y_{Ts} + \mathcal{T}_s, E_s; Z_s)$$

subject to

$$\sum_s \pi_s \left[\mathcal{T}_s \frac{dU^*}{dC^*}(s)\right] = 0$$

Under complete markets the transfer of wealth in each state of the world $\mathcal{T}_s$ is decoupled from monetary policy E_s. This implies the exchange rate has a single role in this economy: closing the output gap, i.e. $V_E(s) = 0$. This is the traditional *demand-management* role of monetary policy. Transfers $\{\mathcal{T}_{cm,s}\}_s$ are then chosen to equalize the marginal utility of tradables abroad and at home, i.e. $\frac{\partial V}{\partial C_T}(s) \propto \frac{dU^*}{dC^*}(s)$.

By contrast, in problem 1 the exchange rate E_s is tightly linked to the transfers $\mathcal{T}_s$ by the relationship $\mathcal{T}_s = (RE_s^{-1} - 1)B$. As a result, the exchange rate plays an additional *insurance* role, given by the desire to replicate the complete-markets transfers $\{\mathcal{T}_{cm,s}\}_s$. When prices are flexible, the planner can perfectly replicate these transfers since the exchange rate plays no demand-management role. When prices are sticky and $\#(S) > 2$, there is a trade-off between both objectives of monetary policy.

2.3 An *almost* linear-quadratic approximation

In the literature on optimal monetary policy, a commonly used technique is to replace the original nonlinear problem with a linear-quadratic (LQ) problem that is valid in a neighborhood of the deterministic steady state. This technique can be applied under very general conditions and provides a locally-valid characterization of the solution up to first order. Problems with a portfolio problem, however, are an exception (Benigno and Woodford, 2012).[9] In appendix A, I use a perturbation approach based on a bifurcation theorem stated in Judd and Guu (2001) to show that one can derive an *almost* LQ (ALQ) problem that provides a valid characterization of the solution around the steady state. Here, I illustrate the approach by applying it to the current setting.

The first step is to derive a linear-quadratic objective around a steady state with an arbitrary portfolio $\bar{B}$. Let bars denote steady-state quantities and lowercase letters denote log deviations from the steady state. I assume that at the steady state $\bar{C}^* = 1$, $\bar{Y}_T = \alpha$ and $\bar{Z} = 1 - \alpha$, which then implies $\bar{R} = \bar{E}^{-1} = 1$ and $\bar{L} = 1$. A second-order approximation of the objective yields

$$\mathcal{W} = \sum_s \pi_s \left\{ (\frac{1+\varphi}{\alpha+\varphi})\alpha c_{Ts} - \frac{(1+\varphi)(1-\alpha)}{2} \left(e_s + \frac{\varphi}{\alpha+\varphi} c_{Ts} - \frac{1+\varphi}{\alpha+\varphi} z_s \right)^2 \right\} + \text{t.i.p.} + \mathcal{O}(\epsilon^3)$$

where t.i.p. stands for "terms independent of policy". To evaluate the linear term in c_{Ts}, one needs to know the behavior of c_{Ts} to second order. A second-order approximation of the budget constraint

[9] At the steady state, all assets are perfect substitutes so the optimal portfolio is indeterminate. Furthermore, since agents are risk neutral to first order, the portfolio is also indeterminate to first order. Formally, this implies that the Jacobian is singular at the steady state, so an implicit function theorem like the one used by Benigno and Woodford (2012) cannot be applied to justify the validity of the LQ approach.

and the foreign no-arbitrage condition yield

$$\alpha c_{Ts} + \frac{1}{2}\alpha c_{Ts}^2 = \alpha y_{Ts} + \frac{1}{2}\alpha y_{Ts}^2 + (r-e_s)\bar{B} + \frac{1}{2}(r-e_s)^2\bar{B} + (r-e_s)\tilde{B} + \mathcal{O}(\epsilon^3)$$
$$\sum_s \pi_s\{(r-e_s) + \frac{1}{2}(r-e_s)^2\} = \sum_s \pi_s\gamma^*(r-e_s)c_s^* + \mathcal{O}(\epsilon^3)$$

where $\gamma^* \equiv -\frac{U^{*\prime\prime}}{U^{*\prime}}$, $\tilde{B} = B - \bar{B}$. Furthermore, note that

$$\sum_s \pi_s(r-e_s)\tilde{B}\bar{R}\bar{E}^{-1} = \mathcal{O}(\epsilon^3),$$

since only the first order behavior of $r-e_s$ is required to evaluate this cross term, $\tilde{B}$ is predetermined, and a first-order expansion of the foreign no-arbitrage condition yields $\sum_s \pi_s(r-e_s) = \mathcal{O}(\epsilon^2)$. This is an important observation, as it implies one does not need to know how B varies with risk to characterize welfare to second order (Samuelson, 1970). Using these observations, the objective becomes

$$\mathcal{W} = -\frac{1}{2}k_0\sum_s \pi_s\left\{\chi\left(\alpha^2 c_{Ts}^2 - 2\alpha\gamma^*(r-e_s)c_s^*\bar{B}\right) + \left(e_s + \frac{\varphi}{\alpha+\varphi}c_{Ts} - \frac{1+\varphi}{\alpha+\varphi}z_s\right)^2\right\} + \text{t.i.p.} + \mathcal{O}(\epsilon^3) \tag{13}$$

where $k_0 = (1+\varphi)(1-\alpha) > 0$ and $\chi = (\alpha+\varphi)^{-1}(1-\alpha)^{-1}\alpha^{-1} > 0$. If B were not a choice variable, then maximizing this objective subject to a first-order approximation of the budget constraint and the foreign no-arbitrage condition,

$$\alpha c_{Ts} = \alpha y_{Ts} + \bar{B}(r-e_s) + \mathcal{O}(\epsilon^2) \tag{14}$$
$$\sum_s \pi_s(r-e_s) = \mathcal{O}(\epsilon^2) \tag{15}$$

would be a proper LQ problem. My result is that solving this problem and maximizing *also* with respect to the steady-state approximation point $\bar{B}$ (a nonlinear but tractable problem) yields a locally-valid approximation of the solution around the steady state. I show this by proving that the first-order conditions of this approximate problem coincide with a perturbation of the first-order conditions of the nonlinear problem. The main advantage of the approximate-problem approach is that there are typically many local solutions to the first-order conditions, since the problem is nonlinear in $\bar{B}$. Keeping track of welfare allows me to check not only whether the solution is a local maximum but also the *best* local maximum among the solutions that stay in a neighborhood of the steady state. Of course, the usual caveats with respect to local approximation methods apply.

In appendix A, I show a general version of this result in the abstract setup of Benigno and Woodford (2012) extended to allow for optimal portfolio choice, which nests the models used in this paper. I prove that, as long as there is a single no arbitrage constraint per asset (i.e. the planner can control portfolios), the approach described above is correct.[10]

Proposition 1. *(ALQ equivalence to perturbation) Maximizing (13) with respect to* $(\{e_s, c_{Ts}\}_s, r)$ *and* $\bar{B}$ *yields a linear approximation of a solution to the first order conditions of problem 1 around* $\epsilon = 0$ *for* $(\{e_s, c_{Ts}\}_{s\in S}, r)$ *and a bifurcation point of the system* $\bar{B}$*.*

Proof. This is a special case of proposition 17 (see appendix A.3). □

3 Optimal policy

In this section, I study the optimal policy in the static model. I start by characterizing the solution when the composite is linear in labor, i.e. $\varphi = 0$, in sections 3.1−3.5. This parametrization implies that the exchange rate that closes the output is independent of the outstanding portfolio position. This simplifies the analysis and serves as a useful stepping stone to understand the main results of the paper. The case $\varphi > 0$ is analyzed in section 3.6.

3.1 Monetary policy focuses on insurance when gross positions are large

Substituting in the constraints (14) and (15) and manipulating (13), the planning problem becomes[11]

$$\max_{\{e_s\}_s, \bar{B}} -\frac{1}{2}\sum_s \pi_s \left(\underbrace{(e_s - e_{dm,s})^2}_{\text{demand management}} + \chi \bar{B}^2 \underbrace{(e_s - e_{in,s}(\bar{B}))^2}_{\text{insurance}} \right) \tag{16}$$

where

$$e_{dm,s} = \frac{1}{\alpha} z_s, \tag{17}$$

is the *demand-management* target, i.e. the exchange rate that closes the output gap, and

$$e_{in,s}(\bar{B}) = -\frac{1}{\bar{B}} \underbrace{(-\alpha y_{Ts} + \alpha \gamma^* c_s^*)}_{=\mathcal{T}_{cm,s}}, \tag{18}$$

is the *insurance* target, i.e. the exchange rate that would replicate complete-markets transfers $\mathcal{T}_{cm,s}$. The objective function is intuitive: It penalizes output gaps, i.e. wedges in production efficiency (the red term), and deviations from complete markets, i.e. wedges in risk sharing (the blue term).

Taking the first-order condition with respect to e_s, I obtain the following result.

[10] Suppose the planner could not tax home-currency bonds. Then, the planning problem would have an additional constraint: the home no-arbitrage condition (4). Such a constraint, however, is to first-order identical to the foreign no-arbitrage condition. In appendix A.4, I show that this implies that there is additional indeterminacy at the steady state: the Lagrange multiplier on one of these no-arbitrage conditions. Thus, one needs to keep track of additional quadratic constraint. Of course, if the optimal tax is 0 in the approximated model, those Lagrange multipliers would be zero. This is the case in this paper when there is an infinitely elastic demand of the home-currency bond by foreigners (see section 3.5).

[11] To simplify the exposition, I assume w.l.o.g. that shocks are mean zero, which implies $r = \mathcal{O}(\epsilon^2)$.

Lemma 2. *(Optimal monetary policy) Consider an economy with small risks, i.e.* $\epsilon \to 0$. *Then,*

$$e_{op,s}(\bar{B}) = \left(1 - \omega(\bar{B})\right) e_{dm,s} + \omega(\bar{B}) e_{in,s}(\bar{B}) + \mathcal{O}(\epsilon^2). \tag{19}$$

where $\omega(\bar{B}) = \frac{\chi \bar{B}^2}{1+\chi \bar{B}^2}$.

The optimal exchange rate is a weighted average of the two exchange-rate targets that reflect the goals of monetary policy. Crucially, the optimal weight ω has two components. First, there is an exogenous component that depends on preferences and technology, controlled by the parameter χ. In this simple example, χ depends on openness, α, since nominal rigidities only affect the nontradable sector.[12] More generally, it depends on the degree of price stickiness, risk aversion, and the elasticity of labor supply, among others. More interestingly, there is an additional *endogenous* component: the portfolio $\bar{B}$. When gross positions $|\bar{B}|$ are large, the planner only needs a small exchange rate movement and, hence, small changes in the output gap, to create a transfer $\mathcal{T}_s$. This follows from equation (18). In other words, providing insurance when $|\bar{B}|$ is large is cheap. On the other hand, it becomes increasingly costly to close the output gap. Doing so requires moving the exchange rate, which implies a large transfer of wealth across borders. Note that the sign of $\bar{B}$ is irrelevant for the argument.

Figure 1 shows the two exchange rate targets and the optimal policy for different levels of $\bar{B}$. The left panel plots the response after a positive innovation to nontradable productivity Z_s. Since the price of nontradables is fixed, the demand management target depreciates to lower the relative price of nontradables and close the output gap (dashed-red line).[13] On the other hand, the insurance target is a peg if $\bar{B} \neq 0$.[14] Any movement in e_s would create a transfer of wealth across borders that is undesirable. The solid-green line plots the optimal exchange rate. The larger $\bar{B}$ is, the closer it is to the insurance target. When $\bar{B} = 0$, there is no insurance role so it coincides with the demand-management target.

The right panel plots the response after a positive innovation to the tradable endowment. The demand-management target is a peg, since the relative price of nontradables does not move in the flexible-price allocation.[15] On the other hand, the insurance target is a hyperbola: when $\bar{B} > 0$ the exchange rate needs to depreciate to create a negative transfer that offsets the tradable endowment shock. The smaller $\bar{B}$ is, the larger the required movement. Again, the optimal exchange rate lies between both targets and is closer to the insurance target as $|\bar{B}|$ increases. The response after c_s^* shocks is analogous.

Lemma 2 is not a complete characterization of the exchange rate policy, since it depends on the portfolio, which is endogenous. I tackle this next.

[12]Note that the relationship is non-monotonic. When α is very high, deviations from demand-management matter little since the non-tradable sector is small. When α is very low, given $\bar{B}$, a small exchange rate change creates a large transfer relative to the size of the tradable sector. Note, however, that the desired transfer and, therefore, $e_{in,s}(\bar{B})$, also become smaller as $\alpha \to 0$.

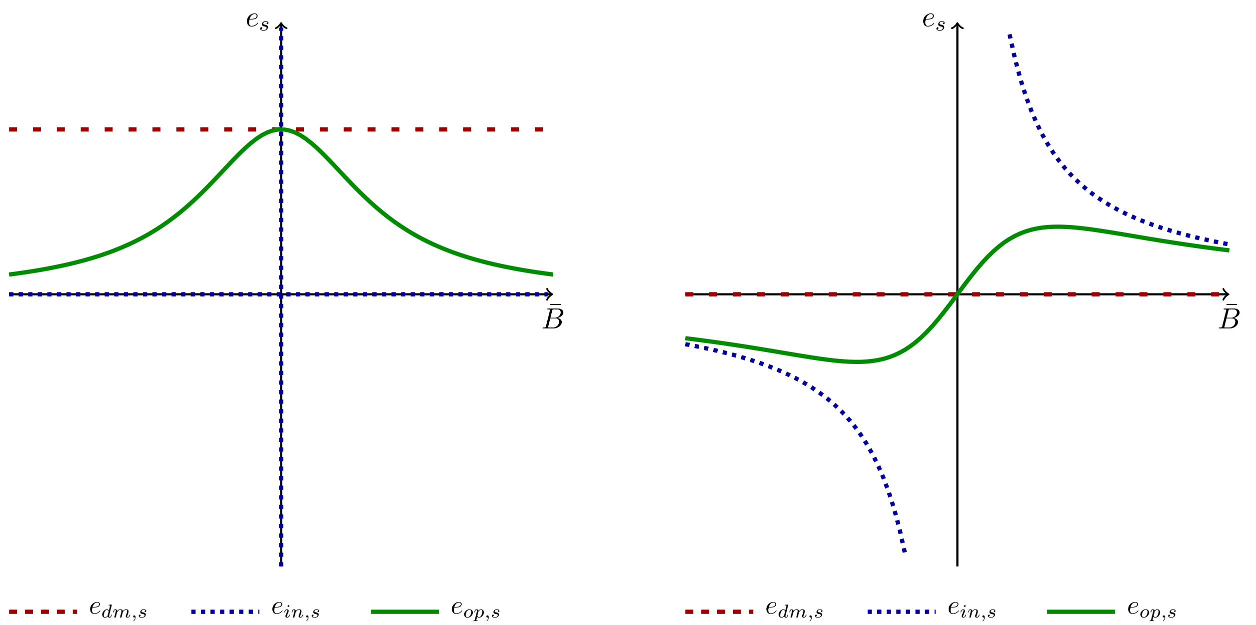
[13]Note that the demand-management target is independent of $\bar{B}$. This because of the unitary elasticity of substitutions and $\varphi = 0$. I relax this in section 3.6.

[14]When $\bar{B} = 0$, there are no transfers of wealth so e_s is irrelevant for risk sharing.

[15]This is because of GHH preferences, unitary elasticities of substitution, and $\varphi = 0$.

Figure 1: Optimal exchange rate conditional on $\bar{B}$

(a) Nontradable productivity shock (b) Tradable endowment shock



Note: Exchange rate response after a positive nontradable productivity shock (left) and a positive endowment shock (right): demand-management target (dashed-red line), insurance target (dotted-blue line), and optimal policy (solid-green line). I set $\alpha = 0.55$ and $\varphi = 0$.

3.2 Optimal portfolio

Next, I characterize the optimal portfolio. Replacing (19) into (16) and rearranging, the planner's problem becomes

$$\max_{\bar{B}} -\frac{1}{2}\frac{\chi k_0}{1+\chi\bar{B}^2}\left(\underbrace{\bar{B}^2\sigma^2_{e_{dm}}}_{\text{demand management}} + \underbrace{\sigma^2_{\mathcal{T}_{cm}}}_{\text{insurance}} + \underbrace{2\bar{B}\sigma_{\mathcal{T}_{cm}e_{dm}}}_{\text{align targets}}\right) \tag{20}$$

where $\sigma^2_{\mathcal{T}_{cm}}$ is the volatility of transfers in the complete markets allocation, $\sigma^2_{e_{dm}}$ is the volatility of the demand-management target, and $\sigma_{\mathcal{T}_{cm}e_{dm}}$ is their covariance, respectively. Solving this problem yields the optimal steady-state portfolio $\bar{B}$, which is one of the solutions to a quadratic equation (in an interior optimum).

The optimal $\bar{B}$ has two important properties, which I describe next.

3.2.1 The planner chooses portfolios to mitigate trade-offs

Choosing the portfolio optimally allows the planner to mitigate the trade offs between insurance and demand management. This effect is captured by the third term in the objective (20). For example, suppose the economy only receives nontradable productivity shocks z_s. By choosing $\bar{B} = 0$, the planner can allow the exchange rate to float freely without creating any undesirable transfers of wealth across borders. Thus, they replicate the behavior of the first-best economy to first order.[16]

When the economy faces more shocks, the planner exploits the correlation between the targets. For example, suppose endowment shocks y_{Ts} and nontradable productivity shocks z_s are positively correlated ($\mathcal{T}_{cm,s}$ and $e_{dm,s}$ are negatively correlated). When the country is long home-currency assets ($\bar{B} > 0$), the targets align: An exchange rate depreciation when z_s is high closes the output gap and creates a negative transfer, which is desirable because the endowment y_{Ts} is high. Thus, by choosing $\bar{B}$ appropriately, the planner can replicate the first-best allocation to first order. When the correlation is imperfect, the first best is unattainable but the same intuition goes through: the planner chooses $\bar{B}$ to align the targets on average.

Proposition 2. *In an interior optimum, the optimal home-currency position $\bar{B}$ has the **opposite sign** to $\sigma_{\mathcal{T}_{cm}e_{dm}}$. If $\mathcal{T}_{cm}$ and e_{dm} are perfectly correlated, the planner attains the first-best allocation to first order (provided $\bar{K}$ is large enough so that the replicating portfolio is feasible).*

3.2.2 When market incompleteness is pervasive, large gross positions are optimal

To the extent that the insurance and demand management motives are not perfectly correlated, the planner needs to prioritize one objective. In section 3.1, I argued that the optimal weight depended on the portfolio $\bar{B}$. When gross positions $|\bar{B}|$ are large, it is relatively cheap to provide insurance and relatively costly to close the output gap. By contrast, small gross positions $|\bar{B}|$ minimize the losses from deviations from demand management $\sigma^2_{e_{dm}}$ (the first term). This argument is reflected in the first and second terms of the objective (20).

This principle guides the optimal portfolio decision when trade-offs are unavoidable: The more important the insurance motive, captured by $\sigma^2_{\mathcal{T}_{cm}}/\sigma^2_{e_{dm}}$ and χ, the larger the gross positions $|\bar{B}|$. In this example economy, $\sigma^2_{\mathcal{T}_{cm}}/\sigma^2_{e_{dm}}$ would be large if y_{Ts} and c^*_s are very volatile relative to z_s. Note that it is crucial that the proper hedges of y_{Ts} and c^*_s are missing securities. If agents could imperfectly hedge these shocks, the volatility of the transfers that need to be replicated $\sigma^2_{\mathcal{T}_{cm}}$ would decrease and the planner would choose a smaller currency exposure (see section 4).

Proposition 3. *In an interior optimum, gross positions $|\bar{B}|$ become **larger** when the insurance motive becomes more important (i.e., when $\sigma^2_{\mathcal{T}_{cm}}/\sigma^2_{e_{dm}}$ or χ increase). Furthermore, a decrease in the covariance between the insurance and the demand-management targets - $|\sigma_{\mathcal{T}_{cm}e_{dm}}|/\sigma^2_{e_{dm}}$ - makes gross positions $|\bar{B}|$ **smaller** if and only if the demand-management motive is more important than the insurance motive, i.e., if $\sigma^2_{e_{dm}} > \chi\sigma^2_{\mathcal{T}_{cm}}$. Conversely, i.e., if $\sigma^2_{e_{dm}} < \chi\sigma^2_{\mathcal{T}_{cm}}$, it makes gross positions $|\bar{B}|$ **larger**.*

3.3 Optimal monetary policy: Risk matters to first order

In standard models, the optimal monetary policy in any given state is independent from the relative likelihood of that state. Here, that is no longer true. The optimal exchange rate response depends

[16] Suppose there are no restrictions on cross-border currency holdings, i.e. $\bar{K} = \infty$. When the economy receives only tradable endowment shocks, the planner can approximate the first-best arbitrarily closely by choosing $\bar{B} \to \infty$ and $e_s \to 0$ such that $\bar{B}e_s = \alpha y_{Ts}$. This case is similar to the one studied by Korinek (2009).

on $\bar{B}$, which in turn depends on the distribution of shocks in the economy. Indeed, an important feature of the solution is that optimal portfolio choice amplifies the bias of optimal monetary policy in favor of one objective. That is, suppose that insurance becomes more important (e.g. $\uparrow \chi$). Proposition 3 implies that the planner chooses larger gross positions, i.e. a larger $|\bar{B}|$. Lemma 2 implies that that both high χ and large gross positions $\bar{B}$ lead to an increase on the optimal weight on the insurance motive ω. In other words, there is not only a direct effect on the weight through χ but also an indirect effect through the optimal $\bar{B}$.

Proposition 4. *The optimal insurance weight ω increases with the importance of the insurance motive (i.e. when $\sigma^2_{\mathcal{T}_{cm}}/\sigma^2_{e_{dm}}$ and χ increase).*

3.4 Portfolio endogeneity is crucial for exchange rate volatility

Should the exchange rate be allowed to "float" to manage aggregate demand or should authorities curb exchange rate volatility? With incomplete markets, letting the exchange rate float to close the output gap is clearly suboptimal, since it may create undesirable transfers of wealth. This may lead to "fear of floating" in some states of the world (e.g. after nontradable productivity shocks Z_s). However, in other states, the planner may actually increase the volatility of the exchange rate to provide insurance (e.g. after tradable endowment shocks Y_{Ts}).

What effect dominates? Using equation (19), one can write exchange rate volatility as a function of the weight on each target and the portfolio,

$$\sigma^2_e(\omega, \bar{B}) = (1-\omega)^2 \sigma^2_{e_{dm}} + \omega^2 \sigma^2_{e_{in}(\bar{B})} + 2\omega(1-\omega)\sigma_{e_{dm}e_{in}(\bar{B})}. \tag{21}$$

The next lemma contains the key observation.

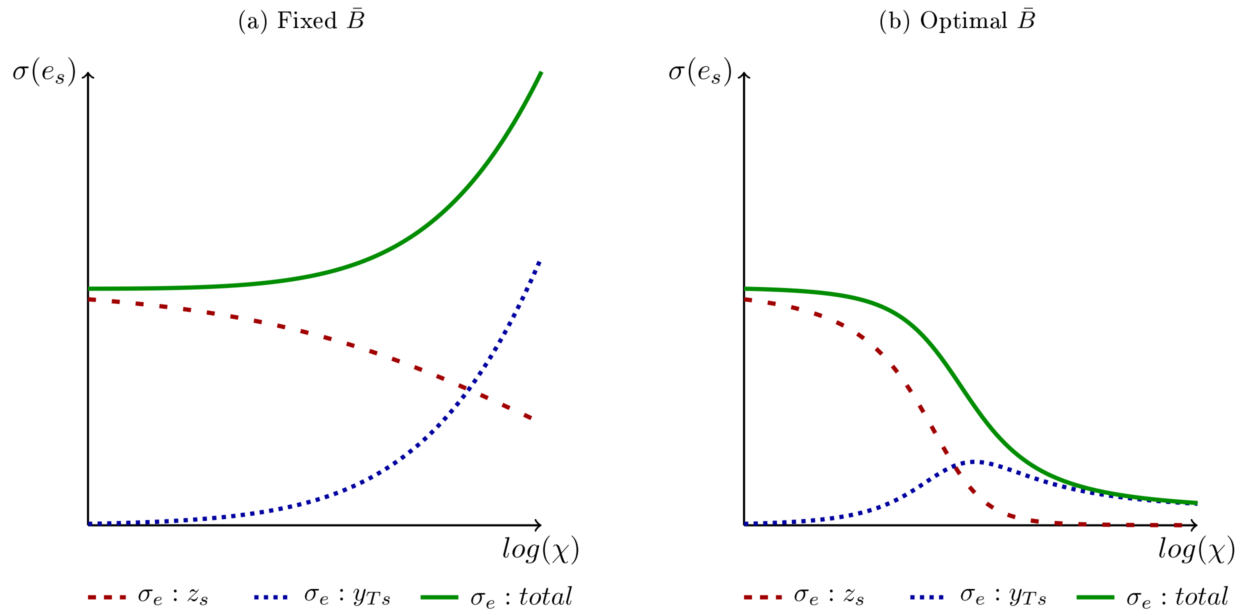
Lemma 3. *In an interior optimum, the solution satisfies $\frac{\partial \sigma^2_e(\omega,\bar{B})}{\partial \omega} = 0$. If gross positions are already at the upper bound, i.e. $|\bar{B}| = \bar{K}$, then $\frac{\partial \sigma^2_e(\omega,\bar{B})}{\partial \omega} > 0$.*

Suppose that the importance of insurance increases (e.g. $\uparrow \chi$). The first effect on volatility comes from the optimal weight ω, which increases (proposition 4). This gives rise to a "composition effect": the exchange rate reacts more to y_{Ts} shocks and less to z_s shocks.

What shock dominates? Figure 2 plots the volatility of the exchange rate explained by z_s shocks (dashed-red line) and the volatility of the exchange rate explained by y_{Ts} shocks (dotted-blue line) shocks as a function of χ. If gross positions $|\bar{B}|$ are already at the upper bound (panel a), i.e. $|\bar{B}| = \bar{K}$, then overall volatility (solid-green line) increases. Intuitively, improving risk sharing is important but large gross positions are infeasible. Thus, the planner must rely on substantial exchange rate movements to create the desired transfers. By contrast, if $\bar{B}$ is at an interior optimum, the planner can also increase $\bar{B}$ to provide insurance. Crucially, lemma 3 states that the planner chooses $\bar{B}$ such that a marginal increase in the weight ω leaves exchange rate volatility unchanged. That is, the composition effect is exactly zero at the solution.

The second effect on volatility comes from the endogeneity of the portfolio. As insurance becomes more important, gross positions $|\bar{B}|$ increase (proposition 2), which reduces the volatility of the

Figure 2: Exchange rate volatility



Note: Variance decomposition of the exchange rate volatility $\sigma(e_s)$ when the portfolio is fixed and too small (left panel) and when the portfolio is optimal (right panel). More precisely, I compute the optimal policy for $\sigma_{y_T} = \sigma_z = 1$, $\sigma_{c^*} = 0$, $corr(y_T, z) = 0.25$, for $\log \chi \in (1, 4)$, which can be rationalized by changing α. I plot exchange rate volatility (i) with only z shocks (dashed-red line), (ii) with only y_T shocks (dotted-blue line), (iii) with both shocks (solid-green line). On the left panel, $\bar{B}$ is fixed at the optimal level for $\log(\chi) = 1$.

insurance target (panel b). Since the composition effect in an interior optimum is zero, exchange rate volatility unambiguously decreases when $\bar{B}$ can adjust (panel b).

Proposition 5. *(Optimal exchange rate volatility). Consider an economy with small risks ($\epsilon \to 0$).*

(i) Suppose gross positions are at the upper bound and, as a result, the optimal portfolio is unresponsive to marginal changes in risks or parameter values (i.e. $|\bar{B}| = \bar{K}$). Then, exchange rate volatility $\sigma_e^2/\sigma_{e_{dm}}^2$ ***increases*** *with the importance of the insurance motive (i.e. when χ and $\sigma_{\mathcal{T}_{cm}}^2/\sigma_{e_{dm}}^2$ increase).*

(ii) Suppose the optimum $\bar{B}$ is interior. Then, exchange rate volatility $\sigma_e^2/\sigma_{e_{dm}}^2$ ***decreases*** *with the importance of the insurance motive (i.e. when χ and $\sigma_{\mathcal{T}_{cm}}^2/\sigma_{e_{dm}}^2$ increase).*

3.5 Portfolio decisions are asymptotically efficient

Does the private sector over- or under-expose itself to home currency debt absent government intervention? Optimality of the portfolio implies

$$\sum_s \pi_s (RE_s^{-1} - 1) \frac{\partial V}{\partial C_T}(s) = 0. \tag{22}$$

Combining a second-order approximation of (22) and (6),

$$\sum_s \pi_s(r-e_s)\left((\frac{\partial\bar{V}}{\partial C_T})^{-1}\frac{\partial V}{\partial C_T}(s)-(\frac{d\bar{U}^*}{dC^*})^{-1}\frac{dU^*}{dC^*}(s)\right)=\mathcal{O}(\epsilon^3). \tag{23}$$

If the portfolio is socially optimal, home's marginal utility of tradables relative to foreign must be uncorrelated with the realized return of the home-currency bond. Combining a second-order approximation of (4) and (6),

$$\sum_s \pi_s(r-e_s)\left((\frac{\partial\bar{V}}{\partial C_T})^{-1}\frac{\partial U}{\partial C_T}(s)-(\frac{d\bar{U}^*}{dC^*})^{-1}\frac{dU^*}{dC^*}(s)\right)=\tau_B+\mathcal{O}(\epsilon^3), \tag{24}$$

where I used that, at the steady state, private and social marginal utilities coincide, and that in an interior optimum the tax is zero at the steady state and to first order (otherwise agents would take infinite positions).

A first-order approximation of $\frac{\partial V}{\partial C_T}(s)$ and $\frac{\partial U}{\partial C_T}(s)$ yields:

$$\underbrace{\frac{\partial V}{\partial C_T}(s)}_{\text{social marginal utility}} = \underbrace{\frac{\partial U}{\partial C_T}(s)}_{\text{private marginal utility}} - \underbrace{\alpha^{-1}(1-\alpha)(e_s-e_s^{dm})}_{\text{aggregate demand externality}} + \mathcal{O}(\epsilon^2). \tag{25}$$

The term in red is an aggregate-demand externality. Agents overvalue tradable consumption in booms: if $e_s > e_{dm,s}$ then $\frac{\partial V}{\partial C_T}(s) < \frac{\partial U}{\partial C_T}(s)$. The opposite occurs in recessions. Since markets are incomplete, the planner will typically deviate from demand management, i.e. $e_s \neq e_{dm,s}$. Therefore, taxes are generically necessary to implement the social optimum (Farhi and Werning, 2016).

Equation (25) illustrates the first crucial assumption behind the approximate zero-tax result: if prices were flexible, there would be no wedges between private and social marginal utility. A common reason why this assumption may be violated in New Keynesian open economy models is terms-of-trade-manipulation motives. For example, in section 4 I show that the tax is not approximately zero if there is a finite elasticity of demand for home assets.

Whether agents over- or under-expose themselves to home-currency risk depends on the correlation of asset returns and output gaps. The key observation is that output gaps in this economy are *purely endogenous*. To understand this, consider a first-order approximation to the first-order condition with respect to the exchange rate in problem 1:

$$e_s-e_{dm,s}\propto(\frac{\partial\bar{V}}{\partial C_T})^{-1}\frac{\partial V}{\partial C_T}(s)-(\frac{d\bar{U}^*}{dC^*})^{-1}\frac{dU^*}{dC^*}(s)-(\eta-\bar{\eta})\bar{\eta}^{-1}+\mathcal{O}(\epsilon^2), \tag{26}$$

where η is the Lagrange multiplier on (6).[17] The planner only allows booms and recessions because they affect the return of the home-currency bond. The latter is valuable when home marginal utility at home diverges from the one abroad, i.e. when risk sharing fails. In other words, if markets were complete, the planner would close output gaps state-by-state. This is the second

[17]More precisely, η is the Lagrange multiplier after a normalization (I divide it by $\bar{B}$).

crucial assumption behind the zero-tax result. Appendix B.5 studies two reasons why standard New Keynesian models may violate this assumption: economies with mark-up shocks and multiple sources of nominal rigidities.[18] In those cases, the optimal approximate tax is generically not zero, even under complete markets.

Putting (26) together with the optimality of the portfolio (23), and using the fact that $\sum_s \pi_s(r - e_s) = \mathcal{O}(\epsilon^2)$ and η is predetermined, it follows that the value of output gaps and realized excess returns must be *uncorrelated* in the approximate solution. That is,

$$\sum_s \pi_s(r - e_s)(e_s - e_{dm,s}) = \mathcal{O}(\epsilon^3).$$

Thus,

$$\sum \pi_s(r - e_s)\left((\frac{\partial \bar{V}}{\partial C_T})^{-1} \frac{\partial U}{\partial C_T}(s) - (\frac{d\bar{U}^*}{dC^*})^{-1} \frac{dU^*}{dC^*}(s) \right) = \mathcal{O}(\epsilon^3) \Leftrightarrow \tau_B = \mathcal{O}(\epsilon^3).$$

The tax is of order ϵ^3 or higher, i.e. it converges to zero faster than the risk premium. For any finite level of risk $\epsilon > 0$, the private portfolio decision is inefficient and taxes are generically nonzero (Farhi and Werning, 2016). However, as risk vanishes $\epsilon \to 0$, the portfolio that the private sector would choose absent taxes and the socially optimal portfolio converge to the same point. In other words, the portfolio is asymptotically efficient.

Proposition 6. *(Asymptotic portfolio taxes) Consider an economy with small risks ($\epsilon \to 0$). Then, in an interior optimum optimal portfolio taxes τ_B are given by*

$$\tau_B = O(\epsilon^3). \tag{27}$$

3.6 Wealth effects

Next, I consider the case of a composite with a strictly convex disutility of labor, i.e. $\varphi > 0$. Following the same steps as in section 3.1, the planning problem becomes

$$\max_{\{e_s\}_s, \bar{B}} -\frac{1}{2}k_0(1 - \mu\bar{B})^2 \sum_s \pi_s \left(\underbrace{(e_s - e_{dm,s}(\bar{B}))^2}_{\text{demand management}} + \chi f(\bar{B})^2 \underbrace{(e_s - e_{in,s}(\bar{B}))^2}_{\text{insurance}} \right) \tag{28}$$

[18] Intuitively, these are cases where the dimension of the aggregate-demand externality is larger than the degrees of freedom of monetary policy so that the planner can only stabilize a weighted average of the externalities. Perhaps surprisingly, appendix B.5 shows that, even allowing for multiple sources of nominal rigidities and mark-up shocks, economies with separable utility between tradables and nontradables feature a "divine coincidence": the weighted average that the planner stabilizes is also the weighted average that matters for the wedge between private and social marginal utility. Therefore, the tax is still approximately zero.

where

$$e_{dm,s}(\bar{B}) = \overbrace{\frac{1}{1-\mu\bar{B}}}^{\text{wealth effect}} \left(\frac{1+\varphi}{\alpha+\varphi}z_s - \mu\alpha y_{Ts}\right)$$

$$e_{in,s}(\bar{B}) = -\frac{1}{\bar{B}}\underbrace{(-\alpha y_{Ts} + \alpha\gamma^* c_s^*)}_{\mathcal{T}_{cm,s}}$$

$$f(\bar{B}) = -\frac{\bar{B}}{\underbrace{1-\mu\bar{B}}_{\text{wealth effect}}}, \text{ with } \mu = \frac{\alpha^{-1}\varphi}{\alpha+\varphi}.$$

Comparing (16) and (28), it becomes clear that the crucial difference between this problem and the one studied in sections 3.1 - 3.5 is that the two objectives of monetary policy are no longer independent. When a transfer makes agents richer ($\mathcal{T}_s > 0$), the exchange rate needs to appreciate to prevent an inefficient boom in the economy. Henceforth, I call this the *wealth effect* of exchange rate movements, which is governed by the parameter μ.

The wealth effect introduces an asymmetry into the optimal weight. For any two positions of the same size, the weight on the insurance objective is larger if agents are long home-currency assets. To see why, suppose the planner wants to create a positive transfer. If agents are short the home currency, the planner needs to depreciate the exchange rate. Since the depreciation makes agents richer, the demand-management target moves in the opposite direction, i.e. it appreciates. This makes the original transfer more costly. Equivalently, suppose the planner was willing to create a deviation of a 1% depreciation with respect to the exchange rate that closes the output gap.[19] If $\bar{B} < 0$, the actual exchange rate movement, and resulting transfer, would be smaller than 1%. Indeed, the additional transfer would be $f(\bar{B})$%.

Lemma 4. *(Optimal monetary policy) Consider an economy with small risks, i.e., $\epsilon \to 0$. Then,*

$$e_s^{op}(\bar{B}) = \left(1-\omega(\bar{B})\right) e_{dm,s}(\bar{B}) + \omega(\bar{B}) e_{in,s}(\bar{B}) + \mathcal{O}(\epsilon^2). \tag{29}$$

where $\omega(\bar{B}) \equiv \frac{\chi f(\bar{B})^2}{1+\chi f(\bar{B})^2}$.

The preceding discussion suggests that the size of the portfolio is not the most adequate measure of "exposure" to monetary policy in this generalized environment, but rather $f(\bar{B})$. Indeed, lemma 2 and proposition 3 hold in terms of $f(\bar{B})$. That is, a larger importance of insurance leads to a larger balance-sheet exposure to monetary policy, $f(\bar{B})$, and an increased weight on the insurance target. Note that, as long as $1-\mu\bar{B} > 0$, a larger exposure $|f(\bar{B})|$ is associated with larger gross positions $|\bar{B}|$.

Henceforth, it will also prove useful to a measure of portfolio returns using $f(\bar{B})$ as a notion of

[19] For a given a transfer $\mathcal{T}_s$, the exchange rate that closes the output gap is $e_s(\mathcal{T}_s) = -\mu\mathcal{T}_s + e_{dm,s}(0)$. For a given deviation $e_s - e_s(\mathcal{T}_s)$, the planner gets an additional transfer of $\mathcal{T}_s - \mathcal{T}_{dm,s}(\bar{B}) = f(\bar{B})(e_s - e_s(\mathcal{T}_s))$, where $\mathcal{T}_{dm,s}(\bar{B}) = f(\bar{B})e_{dm,s}(\bar{B})$ is the implied transfer under demand management.

the size of the portfolio, rr_{fs}:

$$rr_{fs} = \frac{\mathcal{T}_s}{f(\bar{B})}. \tag{30}$$

Proposition 5 holds in terms of rr_{fs}.[20] Note that

$$\frac{rr_{fs}}{rr_{dm,s}(0)} = \frac{rr_s}{rr_{dm,s}(\bar{B})} = \frac{e_s}{e_{dm,s}(\bar{B})}.$$

Thus, relative to a demand-management policy $rr_{dm,s}(\bar{B})$, the volatility of the home-currency returns rr_s decreases with the importance of the insurance objective.

Proposition 7. *Lemma 2 and propositions 3 and 4 hold for* $f(\bar{B}) = -\frac{\bar{B}}{1-\mu\bar{B}}$, $\sigma^2_{\mathcal{T}_{cm}}/\sigma^2_{e_{dm}(0)}$ *and* $\sigma_{\mathcal{T}_{cm}e_{dm}(0)}$ *instead of* $\bar{B}$, $\sigma^2_{\mathcal{T}_{cm}}/\sigma^2_{e_{dm}}$, *and* $\sigma_{\mathcal{T}_{cm}e_{dm}}$, *respectively. Proposition 2 holds in terms of* $f(\bar{B})$ *with the opposite sign, i.e.* $f(\bar{B})$ *has the same sign as* $\sigma_{\mathcal{T}_{cm}e_{dm}(0)}$. *Proposition 5 part (ii) holds for* $\sigma^2_{rr_f}/\sigma^2_{rr_{dm}(0)}$ *instead of* $\sigma^2_e/\sigma^2_{e_{dm}(0)}$, *while part (i) holds as long as* $1 - \mu\bar{B} > 0$.

One may wonder whether the results on $f(\bar{B})$-return volatility translate into the volatility of the actual returns, i.e. the exchange rate. Note that the actual returns, $rr_s = -e_s$, and the $f(\bar{B})$ returns, rr_{fs}, are linked by

$$rr_s = -(\frac{1}{1-\mu\bar{B}})rr_{fs}. \tag{31}$$

The wealth effect implies that the answer depends on the sign of the position. Suppose that there is a positive nontradable productivity shock, which requires a depreciation to close the output gap. When $\bar{B} < 0$, a depreciation makes the country richer, increasing nontradable demand. Thus, the exchange rate needs to depreciate by less to close the output gap, i.e. it becomes less volatile. The opposite is true when $\bar{B} > 0$. Therefore, when $\bar{B} < 0$, which is often the empirically relevant case, this new effect reinforces the effects I characterized in section 3.4. By contrast, when $\bar{B} > 0$, it operates in the opposite direction making the overall effect on volatility ambiguous.

Proposition 8. *(Optimal exchange rate volatility). Consider an economy with small risks* ($\epsilon \to 0$).

(i) Suppose gross positions are at the upper bound and, as a result, the optimal portfolio is unresponsive to marginal changes in risks or parameter values (i.e. $|\bar{B}| = \bar{K}$*). Furthermore, suppose that* $1 - \mu\bar{B} > 0$. *Then, exchange rate volatility* $\sigma^2_e/\sigma^2_{e_{dm}(0)}$ ***increases*** *with the importance of the insurance motive (i.e. when* χ *and* $\sigma^2_{\mathcal{T}_{cm}}/\sigma^2_{e_{dm}(0)}$ *increase keeping* $\sigma_{\mathcal{T}e_{dm}(0)}$ *constant).*

(ii) Suppose the optimum $\bar{B}$ *is interior. Then, if* $\bar{B} < 0$, *exchange rate volatility* $\sigma^2_e/\sigma^2_{e_{dm}(0)}$ ***decreases*** *with the importance of the insurance motive (i.e. when* χ *and* $\sigma^2_{\mathcal{T}}$ *increase). If* $\bar{B} > 0$, *the result is ambiguous.*

Finally, proposition 6 also carries over to this environment. Interestingly, even if the planner focused on demand management, portfolio choices would be approximately efficient. That is, consider problem 1 with the additional constraint that output gaps are zero in every state, i.e.

[20] Note that I write the results relative to $e_{dm}(0)$, which is a function of parameters and shocks, i.e. exogenous.

$E_s^{-1} = E_{dm,s}^{-1} = \Phi(C_{Ts}, Z_s)\ \forall s$, where $\Phi(\cdot)$ is given by

$$\Phi(C_{Ts}, Z_s) \equiv \alpha^{-\frac{\varphi}{\alpha+\varphi}}(1-\alpha)^{\frac{1+\varphi}{\alpha+\varphi}} Z_s^{-\frac{1+\varphi}{\alpha+\varphi}} C_{Ts}^{\frac{\varphi}{\alpha+\varphi}}.$$

The first-order condition with respect to C_{Ts} yields

$$\frac{\partial U}{\partial C_T}(s) - \eta\frac{dU^*}{dC^*}(s) = \left(\Lambda(s) - \eta\frac{dU^*}{dC^*}(s)\right)\left(1 - \underbrace{RB\frac{\partial \Phi}{\partial C_T}(s)}_{\text{pecuniary externality}}\right), \tag{32}$$

where I used the fact that $\frac{\partial U}{\partial C_T}(s) = \frac{\partial V}{\partial C_T}(s)$ when the output gap is zero, i.e. there is no aggregate-demand externality. Still, agents do not internalize that their portfolio decisions affect the return of the asset and, as a result, affect its price. This is a standard pecuniary externality due to incomplete markets (Geanokoplos and Polemarchakis, 1986).

A first-order approximation of (32) yields

$$\frac{\partial U}{\partial C_T}(s) - \bar{\eta}\frac{dU^*}{dC^*}(s) - (\eta - \bar{\eta})\frac{d\bar{U}^*}{dC^*} = \left(1 - \bar{R}\bar{B}\frac{\partial \bar{\Phi}}{\partial C_T}\right)\left(\frac{\partial U}{\partial C_T}(s) - \bar{\eta}\frac{dU^*}{dC^*}(s) - (\eta - \bar{\eta})\frac{d\bar{U}^*}{dC^*}\right) + \mathcal{O}(\epsilon^2).$$

While agents do not internalize that consuming an extra unit in state s changes the realized excess return of the bond by $\bar{R}\frac{\partial\bar{\phi}}{\partial C_T}$, the social value of $\bar{R}\bar{B}\frac{\partial\bar{\phi}}{\partial C_T}$ extra units of tradables is given by social marginal utility. Hence, private and social marginal utilities (relative to foreigners') are still proportional to one another. Using the same argument as before, it follows that pecuniary externalities must also be uncorrelated with the realized excess return of the bond so portfolio decisions are asymptotically efficient. This result is critical to understand why the no-tax result generalizes to arbitrary asset structures in section 4.

Proposition 9. *(Asymptotic portfolio taxes) Consider an economy with small risks ($\epsilon \to 0$). Then, in an interior optimum optimal portfolio taxes τ_B are given by*

$$\tau_B = O(\epsilon^3).$$

Taxes are also approximately zero in any economy where the planner is restricted to set $e_s = e_s^{dm}$ and can only optimize over B.

4 Static model: General framework

The previous analysis makes several strong assumptions. An attractive feature of my methodology is that it can easily accommodate more general environments. In this section, I exploit this tractability to analytically prove the robustness of the previous results and derive additional insights. I defer the analysis of a dynamic economy to section 5. For brevity, a formal definition and characterization of the competitive equilibrium and planning problem are omitted here and included in appendix B.2.

4.1 Set up

Preferences There is a representative agent with preferences over tradables, nontradables, and labor,

$$\sum_s \pi_s U(C_{Ts}, C_{Ns}, L_s; \xi_s). \tag{33}$$

where ξ_s is a $K \times 1$ vector of shocks. U is locally analytic around the steady state, increasing in C_T and C_N, decreasing in L, and strictly concave. Each agent owns an endowment $Y_T(\xi_s)$ of the tradable good and firm profits, described below. I normalize $P^*_{Ts} \equiv 1$.

Technology The nontradable good is a CES composite of a continuum of varieties

$$C_{Ns} = (\int_0^1 C_{Ns}(i)^{\frac{\eta-1}{\eta}} di)^{\frac{\eta}{\eta-1}}.$$

For each variety, there is a firm that produces it using labor,

$$C_{Ns}(i) = F(L_s(i); \xi_s).$$

F is locally analytic around the steady state, increasing and concave.[21] Note that all varieties have the same technology. Thus, in the first best all firms produce equal amounts.

The only role of this special structure is to introduce nominal rigidities into the environment.[22] More precisely, I assume that in each state of the world a random share ϕ of the firms have a fixed home-currency price of $P_{Ns}(i) = 1\ \forall s$ while the remaining share $1-\phi$ can reset their price. As usual, I assume there is a constant production subsidy $\tau^L = 1 - \frac{\eta-1}{\eta}$ to correct the monopolistic distortion.

Financial assets Agents have access to $J+1 \leq K$ assets, $\{\Theta_j\}_{j=0}^J$. For ease of exposition, I assume that one of these assets, labeled asset 0, is a risk-free asset in foreign currency. The payoff $\{\tilde{X}_{js}\}_s$ of asset j may depend on both aggregate endogenous variables, collected in $\mathcal{Y}_s$[23] and shocks ξ_s,

$$\tilde{X}_{js} = \tilde{X}_j(\mathcal{Y}_s; \xi_s)$$

for some function $\tilde{X}_j$ that is positive and locally analytic around the steady state. For example, the home-currency bond of the previous section would be $\tilde{X}_j(\cdot) = E_s^{-1}$. Claims on a mutual fund of nontradable good producers would be $\tilde{X}_j(\cdot) = \int_0^1 \Pi_{Ns}(i) di$ while claims on the tradable endowment would be $\tilde{X}_j(\cdot) = Y_T(\xi_s)$.

[21] To simplify the exposition, I assume production is separable across goods and introduce nonseparabilities (if any) in the utility function. A model where tradables are used in the production of nontradables is similar to one with nonseparable utility.

[22] It is straightforward to, instead, introduce nominal rigidities into retail (i.e. firms that aggregate tradables and nontradables to produce the final good) or wages. As discussed in section 3.5, what is important for the result on taxes (proposition 12) is that if the planner chose to, they can fully undo nominal rigidities (see also appendix B.5).

[23] Here, $\mathcal{Y}_s$ would potentially include tradable and non-tradable consumption, labor, nontradable-goods prices, wages, aggregate profits, and exchange rates. See appendix B.2.1 for a formal definition.

Large economy A measure m of foreigners with endowments $\{Y^*(\xi_s)\}_s$ trades financial assets with home agents. They give rise to a no-arbitrage equation that prices financial assets

$$\sum_s \pi_s \left[(R_j \tilde{X}_j(\mathcal{Y}_s;\xi_s) - 1) \frac{dU^*\left(Y^*(\xi_s) - \frac{1}{m}\sum_j (R_j \tilde{X}_j(\mathcal{Y}_s;\xi_s) - 1)\Theta_j\right)}{dC^*} \right] = 0,$$

where R_j is the equilibrium foreign-currency yield of asset j relative to asset 0 (normalized to one) and Θ_j is the position of the home agent in asset j (financed by issuing asset 0). The case in the previous section is nested by letting $m \to \infty$.

Planner The planner chooses the exchange rate rule $\{E_s\}_s$ and taxes on financial assets $\{\tau_j\}_j$ to maximize

$$\sum_s \pi_s \left\{ U(C_{Ts}, C_{Ns}, L_s; \xi_s) + m\bar{\lambda} U(C_s^*) \right\}$$

subject to all equilibrium conditions (see appendix B.2.2). $\bar{\lambda} \geq 0$ is the relative Pareto weight on foreigners. This formulation nests the case of a home (or non-cooperative) planner with $\bar{\lambda} = 0$ and the case of a global planner (or cooperative) that does not want to redistribute wealth ex ante across borders by setting $\bar{\lambda}$ equal to the ratio of the marginal utility of tradables at the steady state.

4.2 Results

Lemma 8 in appendix B.2.6 shows that the approximate planning problem can be written as

$$\max_{\{e_s\}_s, \{\bar{\Theta}_j\}_j} -\frac{k_0}{2} \left(\frac{1 - \sum_j \mu_j \bar{\Theta}_j}{1 - \sum_j \bar{R}_j \bar{\Theta}_j \frac{\partial X_j}{\partial C_T}} \right)^2 \sum_s \pi_s \left(\underbrace{(e_s - e_{dm,s}(\bar{\Theta}))^2}_{\text{demand management}} + \chi f(\bar{\Theta})^2 \underbrace{(e_s - e_{in,s}(\bar{\Theta}))^2}_{\text{insurance}} \right) \tag{34}$$

where $k_0 > 0$, $\chi > 0$ is the parameter that controls the relative importance of insurance vs. demand-management, $e_{dm,s}(\bar{\Theta})$ is the demand-management target, which attains a zero output gap and zero price dispersion when $\Theta = \bar{\Theta}$, and $e_{in,s}(\bar{\Theta})$ is the insurance target, which replicates the flexible-prices complete-market transfers $\mathcal{T}_{cm,s}$ when $\Theta = \bar{\Theta}$ (explicit expressions provided in the appendix). The balance-sheet exposure to monetary policy $f(\bar{\Theta})$ is given by

$$f(\bar{\Theta}) = \frac{\overbrace{\sum_j k_{rr_j e} \bar{\Theta}_j}^{\text{direct effect}}}{1 - \underbrace{\sum_j \mu_j \bar{\Theta}_j}_{\text{wealth effect}}}.$$

The numerator contains the direct effect: the change in the return of asset j when the exchange rate changes, keeping wealth constant, given by $k_{rr_j e}$. In the model of section 2, $k_{rre} = -1$. The denominator contains the wealth effect: the change in the return of the portfolio because of a transfer $\mathcal{T}_s$, keeping the output gap constant, given by $\sum_j \mu_j \bar{\Theta}_j$. $f(\bar{\Theta})$ has the same interpretation as $f(\bar{B})$ in section 3.6: a depreciation of 1% with respect to the demand-management target creates a transfer of $f(\bar{\Theta})$%.

Written this way, it becomes clear that the optimal exchange rate is once more a weighted average of both targets with an optimal weight that depends on $f(\bar{\Theta})$. However, $\bar{\Theta}$ is now multidimensional and, hence, the optimal exchange rate depends on the entire portfolio $\bar{\Theta}$. To make progress, I divide the optimal portfolio problem into two steps. First, I solve for the optimal portfolio $\bar{\Theta}$ that attains a given level of balance-sheet exposure to monetery policy, $f(\bar{\Theta})$.

Proposition 10. *Let* $\tilde{\Theta}_j = \frac{\bar{\Theta}_j}{1-\sum_j \mu_j \bar{\Theta}_j}$ *and suppose that* $k_{rr_j e} \neq 0$ *for at least one asset* j*. Given some balance-sheet exposure to monetary policy,* $f(\bar{\Theta})$*, the optimal portfolio solves*

$$\tilde{\Theta} = k_{\Theta 0} + k_{\Theta f} f(\bar{\Theta})$$

where

$$k_{\Theta 0} = \left(I - \frac{Var(rr_{dm}(0))^{-1} k_{rre} k'_{rre}}{k'_{rre} Var(rr_{dm}(0))^{-1} k_{rre}} \right) Var(rr_{dm}(0))^{-1} Cov(\mathcal{T}_{cm}, rr_{dm}(0))$$

$$k_{\Theta f} = \frac{Var(rr_{dm}(0))^{-1} k_{rre}}{k'_{rre} Var(rr_{dm}(0))^{-1} k_{rre}},$$

$\mathcal{T}_{cm}$ *are the desired transfers under complete markets,* $k_{rre} = \{kk_{rr_j e}\}_{j=1}^{J} \in \mathbb{R}^{J\times 1}$*, and* $rr_{dm}(0) = \{rr_{j,dm}(0)\}_{j=1}^{J} \in \mathbb{R}^{J\times 1}$ *are the realized excess returns when* $\bar{\Theta} = 0$ *and* $e_s = e_{dm,s}(0)$*. If* $k_{rr_j e} = 0$ $\forall j$*, then* $f(\bar{\Theta}) = 0$ *and*

$$\tilde{\Theta} = Var(rr_{dm}(0))^{-1} Cov(\mathcal{T}_{cm}, rr_{dm}(0)).$$

The first term in the optimal portfolio formula, $k_{\Theta 0}$, captures two effects. First, the planner uses assets with returns that are independent from monetary policy to diversify away some risk. For example, consider the model of section 2 and suppose that the planner can sell claims to the tradable endowment. In this case, the planner can hedge tradable endowment shocks without using monetary policy, which is a costly source of insurance against these shocks. The second effect appears when there is more than one asset that loads on monetary policy. For example, consider the model of section 2 and suppose that the planner can trade an asset that loads on both the tradable endowment shock and the exchange rate. The planner can hedge tradable endowment shocks using this asset and then offset the resulting exposure to monetary policy with an appropriate position on the home-currency bond.

The second term in the optimal portfolio formula, $k_{\Theta f}$, contains information on how the optimal portfolio composition varies with balance-sheet exposure to monetary policy. Interestingly, even if the return of an asset is unaffected by monetary policy, the planner may still vary their holdings of such an asset depending on the desired exposure $f(\bar{\Theta})$. For example, consider the model of section 2 and suppose that the planner can sell claims to the nontradable productivity Z_s. In the original model, if Z_s is very volatile, the planner chooses home-currency bond positions close to 0 to avoid undesirable transfers of wealth. In this example, by contrast, the planner can choose a large B to insure against tradable endowment shocks and buy claims to Z_s to offset the undesirable transfers created by the nominal asset in those states of the world.

This discussion suggests that in this generalized asset market structure, what matters is not the transfers the planner wants to replicate from complete markets $\{\mathcal{T}_{cm,s}\}_s$ or the returns under demand management $\{rr_{dm,s}(0)\}_s$, but rather the transfers that cannot be hedged using instruments other than monetary policy, $\{\tilde{\mathcal{T}}_{cm,s}\}_s$, and the component of returns that depends on the exposure to monetary policy after solving the optimal portfolio problem above, $\{\tilde{rr}_{dm,s}(0)\}_s$. For example, consider the model of section 2. If the planner can trade claims on the tradable endowment, this shock will not enter $\tilde{\mathcal{T}}_{cm,s}$. Similarly, if the planner can trade claims on non-tradable productivity, the exchange rate movements (i.e, the return of the home-currency bond) explained by Z_s will not enter $\tilde{rr}_{dm,s}(0)$. Lemma 5 shows that, once one makes these corrections, the objective function takes the same form as before.

Lemma 5. *The optimal $f(\bar{\Theta})$ solves*

$$\max_{f(\bar{\Theta})} -\frac{1}{2}k_0\left(\frac{\chi}{1+\chi f(\bar{\Theta})^2}\right)\left(\sigma^2_{\tilde{\mathcal{T}}_{cm}}+\sigma^2_{\tilde{rr}_{dm,s}}(0)f(\bar{\Theta})^2-2\sigma_{\tilde{\mathcal{T}}_{cm}\tilde{rr}_{dm,s}}(0)f(\bar{\Theta})\right)+\mathcal{O}(\epsilon^3)$$

where

$$\tilde{\mathcal{T}}_{cm,s}=\mathcal{T}_{cm,s}-k'_{\Theta 0}rr_{dm,s}(0)$$
$$\tilde{rr}_{dm,s}(0)=k'_{\Theta f}rr_{dm,s}(0).$$

It is then immediate that propositions 2, 3, and 4 carry over to this environment.[24] Furthermore, define the $f(\bar{\Theta})$-returns, rr_{fs}, as

$$rr_{fs}=f(\bar{\Theta})^{-1}\tilde{\mathcal{T}}_s, \tag{35}$$

where $\tilde{\mathcal{T}}_s=\mathcal{T}_s-k'_{\Theta 0}rr_{dm,s}(0)$. Intuitively, this object is the transfer created by monetary policy, $\tilde{\mathcal{T}}_s$, per unit of exposure $f(\bar{\Theta})$. I prove an analogous result to proposition 5 for the volatility of $f(\bar{\Theta})$-returns, rr_{fs}. At this level of generality, however, there is no obvious mapping to the volatility of the exchange rate, which needs to be checked on a case-by-case basis.

Proposition 11. *Lemma 2 and propositions 3 and 4 hold for $f(\bar{\Theta})$, $\sigma^2_{\tilde{\mathcal{T}}_{cm}}/\sigma^2_{\tilde{rr}_{dm}(0)}$ and $\sigma_{\tilde{\mathcal{T}}_{cm}\tilde{rr}_{dm}(0)}$ instead of $\bar{B}$, $\sigma^2_{\mathcal{T}_{cm}}/\sigma^2_{e_{dm}}$, and $\sigma_{\mathcal{T}_{cm}e_{dm}}$, respectively. Proposition 2 holds in terms of $f(\bar{\Theta})$ with the opposite sign, i.e. $f(\bar{\Theta})$ has the same sign as $\sigma_{\tilde{\mathcal{T}}_{cm}\tilde{rr}_{dm}(0)}$. Proposition 5 part (ii) holds for $\sigma^2_{rr_f}/\sigma^2_{\tilde{rr}_{dm}(0)}$ instead of $\sigma^2_e/\sigma^2_{e_{dm}(0)}$.*

Finally, proposition 12 characterizes the optimal tax for each asset j.

[24] One may think of $\tilde{rr}_{dm,s}(0)$ and $\tilde{\mathcal{T}}_{cm,s}$ as functions of parameters and shocks that are "sufficient statistics" for the comparative statics emphasized in proposition 11 (see appendix B.2 for details).

Proposition 12. *In an interior optimum, the optimal tax on asset j is given by*

$$\tau_j = \frac{1}{m}\left(1 - \frac{\frac{dU^*}{dC^*}}{\frac{\partial U}{\partial C_T}}\bar{\lambda}\right)\gamma^* Cov(\mathcal{T}_s, rr_{js}) + \mathcal{O}(\epsilon^3).$$

When the country is small, i.e. $m \to \infty$, like in section 2, the tax is zero for every asset in the approximate solution. The reason is that the two crucial assumptions discussed in section 3.5 are satisfied. That is, (i) if prices were flexible, private and social marginal utility would be proportional to one another, and (ii), if markets were complete, the planner would eliminate output gaps. By contrast, when the country is large and policy is non-cooperative, i.e. $m < \infty$ and $\bar{\lambda} \neq \frac{\partial U}{\partial C_T} / \frac{dU^*}{dC^*}$, the first assumption is violated. To see this, suppose the planner only cares about home agents, i.e. $\bar{\lambda} = 0$. The planner realizes that, as the economy demands more insurance from abroad, it becomes more expensive. Private agents do not internalize this negative terms-of-trade externality and over-insure. That is, for any asset that provides insurance for the home country $\text{Cov}(\mathcal{T}_s, rr_{js}) > 0$, the private sector takes a position that is too long and the planner needs to tax this asset $\tau_j > 0$. The opposite is true when a positive position in the asset provides insurance to the rest of the world, i.e. $\text{Cov}(\mathcal{T}_s, rr_{js}) < 0$.[25] This result is reminiscent of a result in Costinot, Lorenzoni and Werning (2014). They show that in a dynamic endowment economy with two countries, a home planner would induce procyclical consumption to manipulate the interest rate in their favor. My result shares the same logic but across states instead of over time, i.e. lack of perfect insurance instead of procyclicality.

5 Dynamic model

In this section, I study a dynamic economy. For ease of exposition, I focus the analysis in sections 5.1−5.4 on a three-period version of the simple model of section 2. Section 5.5 discusses an infinite-horizon version of the general model of section 4, which is studied in detail in appendix B.4.

5.1 Setup

All the uncertainty is still revealed at $t = 1$, but there is an additional period $t = 2$ after it. The utility function is given by

$$\mathcal{W} = \sum_s \pi_s \sum_{t=1,2} \ln\left(\kappa C_{Tst}^{\alpha} C_{Nst}^{1-\alpha} - \frac{1-\alpha}{1+\varphi} L_{st}^{1+\varphi}\right) \tag{36}$$

At $t = 1$, agents can trade one-period home- and foreign-currency bonds that promise a fixed payment at $t = 2$ in the corresponding currency, R_s and R_s^*, respectively. Since the model is

[25] In the special case where agents trade only home- and foreign-currency assets, $\text{Cov}(\mathcal{T}_s, rr_{js}) = \bar{B}\sum_s \pi_s e_s^2$. Thus, the planner pushes positions towards zero, i.e. disincentivizes financial integration.

deterministic between $t = 1$ and $t = 2$, no arbitrage implies

$$R_s^* = E_{s2}^{-1} E_{s1} R_s (1 + \Psi_s), \tag{37}$$

where Ψ_s is a convenience-yield shock, capturing in reduced form unmodelled changes in the liquidity service or pledgeability of home-currency bonds relative to foreign-currency bonds (as in Lahiri and Végh, 2003). When Ψ_s is high, holding the home asset becomes valuable and its price increases. That is, Ψ_s creates a first-order uncovered interest rate parity (UIP) deviation, i.e. Ψ_s is a "UIP shock".[26] I also allow for shocks to R^*, rationalized by fluctuations in foreigners' β^*.[27]

At $t = 0$, agents can trade a long home-currency bond that promises a fixed home-currency coupon of δR_0 at $t = 1$ and $(1-\delta)R_0$ units of the $t = 1$ home-currency bond. In addition, they can trade a short foreign-currency bond that pays 1 unit of foreign currency at $t = 1$.[28] The realized excess return of home-currency bonds at $t = 1$ is given by

$$rr_s = \left\{ \delta \underbrace{\left(R_0 E_{s1}^{-1} - 1\right)}_{\text{affected by } t=1 \text{ policy}} + (1-\delta) \underbrace{\left((1+\Psi_s) R_0 E_{s2}^{-1} R_s^{*-1} - 1\right)}_{\text{affected by } \textbf{promised } t=2 \text{ policy}} \right\}. \tag{38}$$

Equation (38) illustrates the key difference between the dynamic and the static model: the planner has more tools. Before, the only way the planner could create transfers at $t = 1$ was to affect the exchange rate at $t = 1$. Now, they can either move the exchange rate today (i.e. at $t = 1$), or promise to move it tomorrow (i.e. at $t = 2$). The latter would affect the price of home-currency bonds today, i.e. it would create *valuation effects*.

Consumer optimization yields the intratemporal equations (2) and (3) holding $\forall s, t$, a $t = 0$ no-arbitrage condition that is analogous to (4), and an Euler equation,

$$\frac{\partial U}{\partial C_T}(s,1) = (1 - \tau_s^{sav}) R_s^* \frac{\partial U}{\partial C_T}(s,2), \tag{39}$$

where τ_s^{sav} is a savings tax that is uniform across assets. The home country's budget constraint is given by

$$C_{Ts1} + nfa_s = Y_{Ts1} + rr_s B_0 \tag{40}$$

$$C_{Ts2} = Y_{Ts2} + R_s^* nfa_s, \tag{41}$$

where $nfa_s = B_s^* + B_s$ is the net-foreign-asset position. Technology is given by $C_{Nst} = Z_{st} L_{st}$ and nontradables prices are fully rigid and equal to one $P_{Ns}(t) = 1 \ \forall s, t$. Finally, foreign optimization

[26] Note that Ψ_s is symmetric across agents: the liquidity value of the home bond goes up or down for everyone.

[27] The R^* shock I consider has similar effects to what is often labeled "UIP" shock in the literature (Kollmann, 2001). However, in my model R^* does not create a UIP deviation. Here, a positive R^* shock makes *all* assets sold by home agents to foreigners less attractive, regardless of the currency of denomination.

[28] As in section 2, agents only trade assets at $t = 0$ so I normalize w.l.o.g. the yield of the $t = 0$ foreign-currency bond and the $t = 0$ exchange rate to 1.

yields a no-arbitrage condition that is analogous to (6) (see appendix B.3.1 for details). This completes the characterization of the competitive equilibrium.

5.2 Planning problem

The planning problem is to maximize (36) subject to all equilibrium conditions. Appendix B.3.2 shows that a second-order approximation of the objective function yields:[29]

$$\mathcal{W} = -\frac{1}{2}\tilde{k}_0 \sum_s \pi_s \left((1-\alpha)^2 \sum_{t=1,2} \underbrace{x_{st}^2}_{\text{output gap}} + \tilde{\chi} \underbrace{(\mathcal{T}_s - \mathcal{T}_{cm,s})^2}_{\text{insurance}} + 4\tilde{\chi} \underbrace{\tilde{nfa}_s^2}_{\text{savings distortions}} \right) + \text{t.i.p.} + \mathcal{O}(\epsilon^3), \tag{42}$$

where $\tilde{k}_0 = (1+\varphi)(1-\alpha)$, $\tilde{\chi} = \frac{1}{2}\left(\frac{1}{\alpha+\varphi}\right)(1-\alpha)^{-1}\alpha^{-1}$, and

$$\mathcal{T}_{cm,s} = -\alpha y_{s1} - \alpha y_{s2} + \alpha r_s^* + 2\alpha\gamma^* c_s^*$$
$$\tilde{nfa}_s = nfa_s - \underbrace{\frac{1}{2}\left(\alpha(y_{s1} - y_{2s}) + \alpha r_s^* + \mathcal{T}_s\right)}_{= nfa_s^{\text{fb}}(\mathcal{T}_s)}.$$

There are three loss terms. The first two are the same as in the static model: they penalize deviations from production efficiency and perfect risk sharing, respectively. The additional loss term in green reflects that in a dynamic model the planner also cares about the distribution of wealth over time. Let $nfa_s^{\text{fb}}(\mathcal{T}_s)$ denote the first-best savings in an economy that receives an exogenous transfer of $\mathcal{T}_s$. For example, if the country receives a positive transfer of $\mathcal{T}_s$, then under flexible prices the planner would spend $\frac{1}{2}\mathcal{T}_s$ in each period. Whenever savings deviate from this benchmark, consumption smoothing is distorted and welfare decreases.

A first-order approximation of (38) yields

$$rr_s = \delta(r_0 - e_{s1}) + (1-\delta)(\psi_s + r_0 - e_{s2} - r_s^*), \tag{43}$$

where $\psi_s = \log(\Psi_s)$. Using the remaining equilibrium conditions, one can rewrite this expression as a function of output gaps and savings distortions (see appendix B.3.3):

$$-(1-\alpha)(\delta x_{s1} + (1-\delta)x_{s2}) - 2\mu\left(\delta - (1-\delta)\right)\tilde{nfa}_s = \left(1 - \mu\bar{B}\right) rr_s - rr_{dm,s}(0), \tag{44}$$

[29]For ease of exposition, I assume that in the steady state both periods are identical and there is no initial wealth so $\bar{nfa} = 0$. If $\bar{nfa} \neq 0$, then the interest-rate shock has an additional income effect that may be positive or negative depending on whether the country is a creditor or debtor in the original steady state.

where $\mu = \frac{1}{2}\alpha^{-1}\frac{\varphi}{\alpha+\varphi} \geq 0$ is the wealth effect discussed in section 3.6 and[30]

$$rr_{dm,s}(0) = \frac{1}{2}\frac{\varphi}{\alpha+\varphi}(y_{Ts1}+y_{Ts2}) - \frac{1}{2}\frac{\varphi\alpha^{-2}\left(\delta-(1-\delta)\right)}{\alpha+\varphi}r_s^* - \frac{1+\varphi}{\alpha+\varphi}\left(\delta z_{s1}+(1-\delta)z_{s2}\right) + (1-\delta)(\psi_s - r_s^*) \tag{45}$$

are the realized excess returns of the home-currency bond when the planner stabilizes demand ($x_{s1} = x_{s2} = n\tilde{f}a_s = 0$) and $\bar{B} = 0$.

Next, imagine that the planner wants to increase the return of the home-currency bond relative to a laissez-faire demand-management policy. They have three ways of achieving this. First, they can use contractionary monetary policy at $t = 1$, appreciating the nominal exchange rate and creating a recession at $t = 1$, $x_{1s} < 0$. Second, they can rely on forward guidance, promising an expected appreciation of the exchange rate and a future recession, $x_{2s} < 0$. Such a promise would increase the price of home-currency bonds and create a positive valuation effect at $t = 1$. Finally, they can distort consumption. If $\delta > \frac{1}{2}$, the return is more sensitive to the value of the exchange rate today than tomorrow. Thus, by boosting consumption at $t = 1$, i.e. $nfa_s < 0$, they create a positive return without distorting production.

The approximate planning problem is to maximize (42) subject to (44). I solve this problem in two steps. First, I solve the $t = 1$ continuation problem, i.e. finding the optimal combination of output gaps and savings distortions that minimize the cost of a given transfer $\mathcal{T}_s$. Then, I solve the $t = 0$ problem of finding the optimal distribution of realized returns $\{rr_s\}_s$ and the portfolio $\bar{B}$.

5.3 The continuation problem: Minimizing the cost of creating a transfer

Proposition 13 below describes the optimal combination of $\{x_{st}\}_{t=1,2}$ and $n\tilde{f}a_s$. The results are intuitive. First, since the costs of small output gaps and consumption-smoothing distortions are negligible, the planner optimally distorts both margins. When bonds are short, the planner relies more on contemporaneous output gaps and brings consumption forward whenever the goal is a higher home-currency return. When bonds are long, the planner relies on forward guidance to create valuation effects and postpones consumption whenever the goal is a higher home-currency return.

Proposition 13. *Suppose that the planner wants to increase the return of home-currency bonds, i.e.* $rr_s - (1-\mu\bar{B})^{-1}rr_{dm,s}(0) > 0$ *(the converse is analogous). Then:*

(i) Output gaps are negative in both periods. The longer the bonds, the more the planner relies on ***valuation*** *effects, i.e.* $|x_{s1}|$ *increases with* δ *while* $|x_{s2}|$ *decreases with* δ*. When* $\delta = \frac{1}{2}$*, output gaps in both periods are equal.*

(ii) Suppose $\varphi > 0$*. Then, relative to the first-best savings, the planner increases consumption at* $t = 1$ *if* $\delta > \frac{1}{2}$ *(*$n\tilde{f}a_s < 0$*) and decreases consumption at* $t = 1$ *if* $\delta < \frac{1}{2}$ *(*$n\tilde{f}a_s > 0$*). If* $\varphi = 0$ *or*

[30] Note that the wealth effect is smaller in a dynamic economy since consumption increases less than the initial transfer. In an infinite horizon model, $\mu_{\text{dyn}} = (1-\beta)\mu_{\text{static}}$, as agents only spend the annuity value of the transfer.

$\delta = \frac{1}{2}$, *the planner chooses the first-best savings.*

What does this imply for savings taxes? A first-order approximation of the home Euler equation (39) yields

$$c_{Ts2} - (1-\alpha)^2 x_{s2} = r_s^* - \tau_s^{sav} + c_{Ts1} - (1-\alpha)^2 x_{s1} + \mathcal{O}(\epsilon^2), \tag{46}$$

where τ_s^{sav} is a savings tax on home agents (i.e. a financial tax that is uniform across assets). Using the country's budget constraint, (46) can be written as[31]

$$\tau_s^{sav} = \underbrace{(1-\alpha)^2 (x_{s2} - x_{s1})}_{\text{aggregate demand externality}} - \underbrace{2\alpha^{-1} \tilde{nfa}_s}_{\text{pecuniary externality}} + \mathcal{O}(\epsilon^2) \tag{47}$$

Consider first the case with $\varphi = 0$ and suppose that the planner wants to create a positive transfer. When bonds are short (i.e. $\delta > 1/2$), the recession at $t = 1$ is deeper than the one at $t = 2$. Seeing relatively high prices of nontradables at $t = 1$, home agents mistakenly consume *too little* at $t = 1$. The planner needs to tax savings to correct this aggregate-demand externality. The converse is true if $\delta < 1/2$.

Next, suppose that $\varphi > 0$. When bonds are short (i.e. $\delta > 1/2$), the planner wants to consume more at $t = 1$ than under flexible prices to appreciate the exchange rate, i.e. $\tilde{nfa}_s < 0$. However, agents do not internalize the effect of their decisions on the exchange rate. The planner needs to tax savings to correct this pecuniary externality. The converse is true if $\delta < 1/2$.

Proposition 14. *Suppose that the planner wants to boost the return of home-currency bonds, i.e.* $rr_s - (1-\mu\bar{B})^{-1} rr_{dm,s}(0) > 0$ *(the converse is analogous). If* $\delta > \frac{1}{2}$ *(shorter bonds), the planner taxes savings* $\tau_s^{sav} > 0$*. If* $\delta < \frac{1}{2}$ *(longer bonds), the planner subsidizes savings* $\tau_s^{sav} > 0$*.*

5.4 Time-zero problem

After replacing the solution of the continuation problem into (42),[32] the objective becomes

$$\mathcal{W} = -\frac{1}{2} k_0 (1-\mu\bar{B})^2 \sum_s \pi_s \left\{ \underbrace{\left(rr_s - rr_{dm,s}(\bar{B})\right)^2}_{\text{demand management}} + \chi f(\bar{B})^2 \underbrace{\left(rr_s - rr_{in,s}(\bar{B})\right)^2}_{\text{insurance}} \right\} + \text{t.i.p.} + \mathcal{O}(\epsilon^3). \tag{48}$$

[31] In appendix B.4.15, I characterize these taxes in a dynamic version of the general model of section 4. I show that one can still decompose them into the same two components. Interestingly, the sign of the aggregate-demand externality term depends on the utility function. While GHH always predicts that agents overvalue tradables in booms, even with non-unitary elasticities, when labor is separable the sign depends on whether goods are Edgeworth substitutes (agents overvalue tradables in booms) or complements (agents undervalue tradables in booms). Relatedly, Bianchi and Coulibaly, 2022 show that, when monetary policy is used to manipulate savings, whether it leans with or against the wind also depends on whether goods are Edgeworth complements or substitutes.

[32] See appendix B.3.2 for the explicit solution.

where $k_0 = (\delta^2 + (1-\delta)^2 + \tilde{\chi}^{-1}\mu^2(\delta - (1-\delta))^2)^{-1}\tilde{k}_0$, $\chi = (\delta^2 + (1-\delta)^2 + \tilde{\chi}^{-1}\mu^2(\delta - (1-\delta))^2)\tilde{\chi}$, and

$$rr_{dm,s}(\bar{B}) = \frac{1}{1-\mu\bar{B}} rr_{dm,s}(0)$$
$$rr_{in,s}(\bar{B}) = \frac{1}{\bar{B}}\mathcal{T}_{cm,s}.$$

This problem has exactly the same form as in (28) in terms of the realized returns rr_s instead of the exchange rate e_s. Therefore, it is straightforward to show that propositions 7 and 8 carry over to this environment cast in terms of the returns of the home-currency bond. Perhaps more surprisingly, proposition 9 also carries over. The key observation in the proof is that all distortions, e.g. output gaps and savings distortions, are proportional to the required deviation from demand management; this follows from the solution of the continuation problem. As a result, they are all orthogonal to the return of the asset.

Why are portfolio decisions approximately efficient while savings decisions are not? To establish the asymptotic optimality of the private portfolio, it is enough to show that the relative strength of the pecuniary and aggregate-demand externalities across states is proportional to the value of improving insurance. By contrast, the relative strength of these externalities over time is driven by technological features that determine the most cost-effective way of improving insurance, e.g. if the bond is short, then it is better to create an output gap today than in the future. This is not internalized by private agents and, hence, savings taxes are a useful additional tool.

In sum, the additional complexity of this model is encoded in the sufficient statistics $\{rr_{dm,s}(0)\}_s$ and $\{\mathcal{T}_{cm,s}\}_s$ and the parameters χ and μ. For example, convenience-yield shocks ψ_s lead the planner to use exchange rates to stabilize home-currency returns, i.e. they increase $\sigma^2_{rr_{dm}(0)}$ without changing $\sigma^2_{\mathcal{T}_{cm}}$. This pushes positions towards zero. On the other hand, interest rate shocks create a demand for insurance, increasing $\sigma^2_{\mathcal{T}_{cm}}$. Since they also affect returns, there is a natural covariance $\sigma_{rr_{dm}(0)\mathcal{T}_{cm}}$ that pushes the economy towards a particular level of $\bar{B}$. Indeed, if bonds are not too long, a positive interest rate shock $r^*_s > 0$ creates a demand for insurance $\mathcal{T}_{cm,s} > 0$ and lower home-currency returns under demand management $rr_{dm,s}(0) < 0$. This pushes the home economy to be short home-currency bonds.

Proposition 15. *Lemma 2 and propositions 2, 3 and 4 hold for* $f(\bar{B})$, $\sigma^2_{\mathcal{T}_{cm}}/\sigma^2_{rr_{dm}(0)}$, *and* $\sigma_{\mathcal{T}_{cm}rr_{dm}(0)}$ *instead of* $\bar{B}$, $\sigma^2_{\mathcal{T}_{cm}}/\sigma^2_{e_{dm}}$, *and* $\sigma_{\mathcal{T}_{cm}e_{dm}}$, *respectively. Let* $rr_f = f(\bar{B})^{-1}\mathcal{T}_s$. *Proposition 5 part (ii) holds for* $\sigma^2_{rr_f}/\sigma^2_{rr_{dm}(0)}$ *instead of* $\sigma^2_e/\sigma^2_{e_{dm}(0)}$. *Proposition 8 holds for* $\sigma^2_{rr_s}/\sigma^2_{rr_{dm}(0)}$ *instead of* $\sigma^2_e/\sigma^2_{e_{dm}(0)}$.

Proposition 16. *Consider an economy with small risks* ($\epsilon \to 0$)*. Then, in an interior optimum optimal portfolio taxes* τ_B *are given by*

$$\tau_B = O(\epsilon^3).$$

Taxes are also approximately zero in any economy where the planner is restricted to set $e_s = e^{dm}_s$ *and/or* $\tilde{nfa}_s = 0$.

5.5 Evolving uncertainty and infinite horizon

One may wonder whether the previous results were driven by the finite horizon of our economy and the fact that there was no further uncertainty after $t = 1$. In appendix B.4, I consider an infinite-horizon version of the model in section 4 with Calvo pricing. To keep the environment stationary, I assume that each asset j pays a coupon that declines at a geometric rate δ_j, as in e.g. Hatchondo and Martinez (2009).

Most results generalize to this environment (see appendixes B.4.11, B.4.12 and B.4.14 for formal statements). The only caveat at this level of generality is that there is no one-dimensional sufficient statistic $f(\bar{\Theta})$ when there are multiple assets with returns that are endogenous to policy. For example, suppose there are two home-currency bonds traded at $t = 0$: one pays at $t = 1$ and the other pays at $t = 2$. Then, "exposure" to monetary policy is a two-dimensional object. For this reason, there is no analogue of proposition 11.[33] By contrast, when there are multiple assets but only one of them loads on monetary policy, e.g. a model with home-currency bonds and other assets that load on shocks ξ, then this result also carries over. Finally, the result on optimal taxes is also robust, regardless of the number of assets with returns that are endogenous to policy. When $m \to \infty$ or $\bar{\lambda} = (\partial U/\partial C_T)/(\partial U/\partial C^*)$, the two crucial assumptions discussed in section 3.5 are satisfied and, therefore, the optimal tax is approximately zero. Otherwise, the planner taxes financial assets to reduce cross-border insurance and manipulate the terms of trade.

Why are the results so general? The critical observation is that uncertainty more than one-period ahead does not matter for decisions today because utility is quadratic in the approximate model. That is, even though there is uncertainty for $t \geq t_0 + 1$, one can still write a deterministic continuation problem for expectations conditional on t_0 information, as in section 5.3. After solving this, finding the optimal distribution of realized returns at t_0 and the $t_0 - 1$ portfolio is a problem that is isomorphic to the one in the static model.

6 Numerical illustration

In this section, I calibrate an infinite-horizon version of the economy in section 5 with Calvo pricing in the nontradable sector and flow utility function

$$u(C_{Tt}, C_{Nt}, L_t) = \frac{1}{1-\gamma}\left(\left(\alpha^{\frac{1}{\rho}} C_{Tt}^{\frac{\rho-1}{\rho}} + (1-\alpha)^{\frac{1}{\rho}} C_{Nt}^{\frac{\rho-1}{\rho}}\right)^{\frac{\rho}{\rho-1}} - \frac{1-\alpha}{1+\varphi} L_t^{1+\varphi}\right)^{1-\gamma}$$

Table 1 presents the baseline calibration, based on data from Canada, which I take as a benchmark advanced small open economy.

The top panel contains the parameter values that govern preferences, technology, and the be-

[33] The model is still easy to solve numerically, however. In appendix B.4.13, I provide a solution method for this case.

Table 1: Parameter values and shocks

Parameter	Description	Value	Parameter	Description	Value
		A. Structural parameters			
β	Discount factor	0.99	ϕ	Probability of not adjusting prices	0.75
γ	Home risk aversion	2	η	Elasticity of substitution (varieties)	6
γ^*	Foreign risk aversion	2	δ	Bond depreciation	0.042
α	Tradable share	0.55	ϕ_π	Reaction to inflation	2.77
φ^{-1}	Frisch elasticity	0.5	ϕ_x	Reaction to output gap	1.15
ρ	Elasticity of substitution (T/NT)	0.74	ρ_i	Smoothing coefficient	0.87
		B. Shocks			
σ_z	Productivity s.d.	0.71%	ρ_ψ	Convenience yield persistence	0.85
σ_{p*}	Terms-of-trade s.d.	0.2%	$corr(\epsilon_t^z, \epsilon_t^{p*})$	Correlation: z and p^*	0.38
σ_{r*}	World interest-rate s.d.	0.22%	$corr(\epsilon_t^z, \epsilon_t^{r*})$	Correlation: z and r^*	-0.18
σ_{y*}	Foreigners' output s.d.	0.56%	$corr(\epsilon_t^z, \epsilon_t^{y*})$	Correlation: z and y^*	0.56
σ_ψ	Convenience yield s.d.	1.05%	$corr(\epsilon_t^{p*}, \epsilon_t^{r*})$	Correlation : p^* and r^*	-0.50
ρ_z	Productivity persistence	0.85	$corr(\epsilon_t^{p*}, \epsilon_t^{y*})$	Correlation: p^* and y^*	0.44
ρ_{p*}	Terms-of-trade persistence	0.75	$corr(\epsilon_t^{r*}, \epsilon_t^{y*})$	Correlation: r^* and y^*	-0.22
ρ_{r*}	World interest-rate persistence	0.85	$corr(\epsilon_t^{\psi}, \epsilon_t^{x})$	Correlation: ψ and others	0
ρ_{y^*}	World output persistence	0.90			

havior of the Central Bank in the competitive equilibrium, described by a standard Taylor rule,

$$i_t = \rho_i i_{t-1} + (1-\rho_i)\phi_\pi \pi_{Nt} + (1-\rho_i)\phi_\pi x_t,$$

where π_{Nt} and x_t are the welfare-relevant inflation and output gap, respectively. Most of these parameters take values that are standard in the literature. The tradable share α, the bond duration δ, and the Taylor-rule parameters are specific to Canada (see appendix C.1 for details).

The bottom panel contains the parameter values that govern the stochastic processes of the structural shocks. The tradable endowment shock Y_{Tt} is decomposed into two parts: TFP Z_{Tt} and the terms-of-trade P^*_{Tt} so that $Y_{Tt} = P^*_{Tt} Z_{Tt}$. I assume productivity shocks Z_{Tt} and Z_{Nt} are perfectly correlated across sectors due to lack of data on sectoral output at a quarterly frequency. In addition, the economy faces foreign-interest-rate shocks R^*_t and foreign SDF shocks Y^*_t. To match these, I fit AR(1) processes to (log) labor productivity (z), the (log) terms of trade in Canada (p^*), (log) U.S. real seasonally-adjusted output (y^*) and the U.S. 3 month treasury bill rate deflated by the U.S. CPI (r^*), using quarterly HP-filtered data (except for r^*) over the sample period $1997:1-2019:4$. Since it matters for optimal portfolios, I take into account the contemporaneous correlation in the innovations of these AR(1) processes. Finally, I assume that there are shocks to the convenience yield of home bonds relative to foreign bonds, ψ_t, which I assume is independent from other shocks. I choose the volatility of this shock to match the observed home-currency position against the rest-of-the world (about 30% over yearly GDP - Bénétrix et al., 2019).[34]

[34]The persistence of this shock is irrelevant for the results (one can always offset a higher persistence with smaller innovations), so I assume it is the same as the interest-rate shock w.l.o.g.

Table 2: Mapping to sufficient statistics

	$\sigma(\mathcal{T}_{cm})$	$\sigma(rr_{dm}(0))$	$corr(\mathcal{T}_{cm}, rr_{dm}(0))$
Productivity (z)	3.02%	0.27%	1
Terms-of-trade (p^*)	0.43%	0.01%	-1
Foreign interest rate (r^*)	43.50%	1.26%	-1
Foreign SDF (y^*)	35.45%	0%	undefined
Convenience-yield shock (ψ)	0%	6.40%	undefined
Total	49.02%	6.52%	-0.15

Note: The volatility of transfers is measured as a % of GDP.

6.1 Main results

Table 2 shows how much each shock contributes to the variance of the two "sufficient statistics" - the demand for insurance $\mathcal{T}_t$ - and the volatility of the realized excess returns without home-currency bonds $rr_{dm}(0)$, as well as the correlation between them induced by each shock. The demand for insurance is largely driven by the two foreign shocks, r^* and y^*. Realized returns, on the other hand, are mainly driven by the convenience-yield shock. Since the interest rate shock r^* introduces a negative correlation between returns and desired transfers, while ψ and y^* do not introduce any correlation, the optimal country portfolio is short home-currency assets.

Table 3 compares the behavior of the economy under four different policies: (i) the competitive equilibrium with a Taylor rule; (ii) a demand-management policy, i.e. setting $\omega = 0$; (iii) the optimal policy; (iv) the optimal monetary policy if the portfolio were fixed at the calibrated value. The results suggest that even under the optimal policy stabilizing demand is the most important role of monetary policy with an optimal weight of 89% (first row). However, monetary policy still plays an important risk-sharing role: it significantly improves the insurance properties of the home-currency bond, more than doubling the optimal size of gross positions (second row). The endogeneity of the portfolio is crucial: If the portfolio were fixed, the weight on the insurance motive would be about seven times smaller. As argued in section 3.4, by increasing gross positions, the optimal policy manages to improve insurance while lowering the volatility of realized returns (third row). By contrast, with a fixed portfolio, return volatility increases. The second panel computes the volatility of the returns induced by each type of shock. Here, one can appreciate that the composition effect discussed in the theory is quantitatively important: returns move less after convenience-yield shocks and productivity shocks, stabilizing asset returns at the expense of fluctuations in aggregate demand, and move more when the demand for insurance arises (p^*, r^* and y^* shocks).

Table 3: Results in baseline model.

	Taylor rule	Demand management	Optimal	Optimal: fixed Θ
A. Optimal weight, optimal portfolio and volatility of the realized excess returns				
ω		0%	11.37%	1.66%
$\bar{\Theta}$	−30.00%	−28.97%	−85.83%	−30.00%
$\sigma(rr)$	6.40%	6.39%	5.91%	6.41%
B. Variance decomposition of realized excess returns				
$\sigma(rr) : z$	0.07%	0.26%	0.12%	0.22%
$\sigma(rr) : p^*$	0.01%	0.01%	0.02%	0.01%
$\sigma(rr) : r^*$	1.33%	1.23%	2.49%	1.81%
$\sigma(rr) : y^*$	0%	0%	1.17%	0.49%
$\sigma(rr) : \psi$	6.26%	6.27%	5.34%	6.16%
C. Welfare gains (% of first-best)				
Welfare gains	1.32%	2.28%	6.49%	3.90%

Note: In column 4, the portfolio is fixed at −30.0% while the remaining columns it is optimally chosen by the planner. The portfolio is normalized by annual gdp. Every other variable is expressed in quarterly units. Welfare gains are measured by how much of the welfare gap between the first-best (a model with flexible prices) and an economy without home bonds ($\bar{B} = 0$) economy is achieved by each policy: $\frac{welfare(policy) - welfare(\bar{B}=0)}{welfare(first\ best) - welfare(\bar{B}=0)}\%$.

How effective is the optimal policy in completing markets? To answer this question, I compute the welfare gains (in consumption equivalents) of moving from an economy without home-currency bonds to an economy with these bonds and flexible prices - an economy that may be called "first-best" since it can deliver as much insurance as the planner desires at no efficiency cost. I then compute what share of these gains are attained in the economy with sticky prices under each policy. The bottom panel shows the results. The optimal policy improves significantly over the demand-management policy, especially when the portfolio is allowed to adjust. In other words, both optimal monetary policy and optimal portfolio choice are important to maximize the welfare gains of financial integration.

6.2 Additional results

Appendix C uses the calibrated model to conduct additional exercises. First, I show that the quantitative relevance of the insurance channel is sensitive to the parameters that govern the importance of the demand-management motive (ϕ, η, and φ). Second, I deviate from the assumption of small open economy by allowing for a finite measure of foreigners m. I quantify the optimal tax that the home planner sets, show that this significantly decreases gross positions, and contrast the solution with that of a cooperative planner. Third, I compute the optimal policy when the planner cannot use capital controls (i.e. no taxes are allowed). I show that the savings taxes discussed in section 5 do not play a quantitatively relevant role in the baseline calibration, but that this result is sensitive to the maturity of the home-currency bond. In particular, they become more important when bonds are shorter. This is in line with proposition 14, which suggests that savings taxes are more effective when the asset promises payments that are uneven over time.[35] Furthermore, I show that it is easier for the planner to provide insurance with long maturity bonds, since they can make promises farther into the future when prices have had time to adjust. Finally, I show that the optimal insurance weight, gross positions and the welfare gains of financial integration increase in the openness of the economy (α), decrease with the elasticity of substitution between tradables and nontradables (ρ), increase with risk aversion (γ), and the discount factor (β^{-1}).

7 Conclusion

I developed a framework to study optimal monetary policy and capital controls in open economies with incomplete markets and portfolio choice. Optimal monetary policy is a weighted average of two targets: a demand-management target, concerned with the traditional role of "undoing" nominal rigidities, and an insurance target, concerned with improving international risk sharing.

I showed three main results, which underscore the importance of modelling the portfolio decision. First, I showed that the planner chooses optimal portfolios to minimize the ex-post trade-off between both objectives of monetary policy. When the trade-off is unavoidable, the planner chooses larger gross positions when insurance considerations become more important. Second, I showed that when there is a larger need for insurance, the stochastic properties of the realized returns of home-currency assets change via a composition effect (returns move more to improve insurance and less to stabilize demand) and an endogenous portfolio effect (gross positions increase, which changes the volatility of the targets). The importance of each channel and the overall effect on return volatility critically depends on whether the portfolio is allowed to adjust. Finally, I showed that portfolio decisions are approximately efficient despite the presence of aggregate demand externalities (due to nominal rigidities) and pecuniary externalities (due to incomplete markets), so no capital controls on the composition of capital flows are necessary in the approximate solution, as long as the country cannot affect the foreigners' stochastic discount factor. By contrast, savings taxes, i.e. uniform taxes across asset classes, are desirable.

In this paper, I focused on the trade-off between insurance and demand-management, abstracting from other relevant macroeconomic forces such as endogenous terms-of-trade movements in product markets, investment, and financial frictions. However, the methodology I develop is widely

[35] In the simple model, the relevant benchmark was $\delta = \frac{1}{2}$, but this was only because there were only two periods and no discounting. In the infinite horizon setting, as δ goes to zero, payments become more evenly spread out over time.

applicable and can be used to explore optimal policy with portfolio choice in those environments as well. Furthermore, the tools developed in these paper could also be interesting to study closed economies with heterogeneous agents that make portfolio choices with aggregate risk.

References

Akinci, Özge. 2011. "A Note on the Estimation of the Atemporal Elasticity of Substitution Between Tradable and Nontradable Goods." *Mimeo*, 1–17.

Bénétrix, Agustín, Deepali Gautam, Luciana Juvenal, and Martin Schmitz. 2019. *Cross-border currency exposures.* International Monetary Fund.

Benigno, Pierpaolo. 2009*a*. "Are valuation effects desirable from a global perspective?" *Journal of Development Economics*, 89(2): 170–180.

Benigno, Pierpaolo. 2009*b*. "Price Stability with Imperfect Financial Integration." *Journal of Money, Credit and Banking*, 41(1).

Benigno, Pierpaolo, and Michael Woodford. 2012. "Linear-quadratic approximation of optimal policy problems." *Journal of Economic Theory*, 147(1): 1–42.

Bianchi, Javier, and Louphou Coulibaly. 2022. "Liquidity Traps, Prudential Policies, and International Spillovers." National Bureau of Economic Research.

Burstein, Ariel T, Joao C Neves, and Sergio Rebelo. 2003. "Distribution costs and real exchange rate dynamics during exchange-rate-based stabilizations." *Journal of monetary Economics*, 50(6): 1189–1214.

Chang, Roberto, and Andrés Velasco. 2006. "Currency mismatches and monetary policy: A tale of two equilibria." *Journal of International Economics*, 69(1 SPEC. ISS.): 150–175.

Corsetti, Giancarlo, Luca Dedola, and Sylvain Leduc. 2010. "Optimal monetary policy in open economies." In *Handbook of International Economics, Vol. III.*

Costinot, Arnaud, Guido Lorenzoni, and Iván Werning. 2014. "A Theory of Capital Controls as Dynamic Terms-of-Trade Manipulation." *Journal of Political Economy*, 122(1): 77–128.

Devereux, Michael B, and Alan Sutherland. 2007. "Monetary policy and portfolio choice in an open economy macro model." *Journal of the European Economic Association*, 5(2).

Devereux, Michael B., and Alan Sutherland. 2008. "Financial globalization and monetary policy." *Journal of Monetary Economics*, 55(8): 1363–1375.

Devereux, Michael B., and Alan Sutherland. 2011. "Country portfolios in open economy macro-models." *Journal of the European Economic Association*, 9(2): 337–369.

Drenik, Andres, Rishabh Kirpalani, and Diego J Perez. 2022. "Currency choice in contracts." *The Review of Economic Studies*, 89(5): 2529–2558.

Du, Wenxin, Carolin E Pflueger, and Jesse Schreger. 2020. "Sovereign debt portfolios, bond risks, and the credibility of monetary policy." *The Journal of Finance*, 75(6): 3097–3138.

Engel, Charles, and JungJae Park. 2022. "Debauchery and original sin: The currency composition of sovereign debt." *Journal of the European Economic Association*, 20(3): 1095–1144.

Evans, Martin D D, and Viktoria Hnatkovska. 2012. "A method for solving general equilibrium models with incomplete markets and many financial assets." *Journal of Economic Dynamics and Control*, 36(12): 1909–1930.

Farhi, Emmanuel, and Iván Werning. 2016. "A Theory of Macroprudential Policies in the Presence of Nominal Rigidities." *Econometrica*, 84(5): 1645–1704.

Gali, Jordi, and Tommaso Monacelli. 2005. "Monetary Policy and Exchange Rate Volatility in a Small Open Economy." *The Review of Economic Studies*, 72(3): 707–734.

Geanokoplos, John D., and Heraklis M. Polemarchakis. 1986. "Existence, regularity, and constrained suboptimality of competitive allocations when the asset market is incomplete."

Gourinchas, Pierre-olivier, and Hélène Rey. 2013. "External Adjustment, Global Imbalances and Valuation Effects." In *Handbook of International Economics.* 585–640.

Hatchondo, Juan Carlos, and Leonardo Martinez. 2009. "Long-duration bonds and sovereign defaults." *Journal of International Economics*, 79(1): 117–125.

Judd, Kenneth L., and Sy-Ming Guu. 2001. "Economic Theory." *Economic Theory*, 18(1): 127–157.

Kollmann, Robert. 2001. "The exchange rate in a dynamic-optimizing business cycle model with nominal rigidities: A quantitative investigation." *Journal of International Economics*, 55(2): 243–262.

Korinek, Anton. 2009. "Insurance properties of local and foreign currency bonds in small open economies." *Manuscript, Johns Hopkins University.*

Lahiri, Amartya, and Carlos A Végh. 2003. "Delaying the Inevitable : Interest Rate Defense and Balance of Payments Crises Delaying the Inevitable : Interest Rate Defense and Balance of Payments Crises Amartya Lahiri Carlos A . Ve." *Journal of Political Economy*, 111(2): 404–424.

Lane, Philip R., and Gian Maria Milesi-Ferretti. 2007. "The external wealth of nations mark II: Revised and extended estimates of foreign assets and liabilities, 1970-2004." *Journal of International Economics*, 73(2): 223–250.

Lane, Philip R., and Jay C. Shambaugh. 2010. "The long or short of it: Determinants of foreign currency exposure in external balance sheets." *Journal of International Economics*, 80(1): 33–44.

Lustig, Hanno, Christopher Sleet, and Sevin Yeltekin. 2008. "Fiscal hedging with nominal assets." *Journal of Monetary Economics*, 55(4): 710–727.

Mehra, Rajnish, and Edward C. Prescott. 1985. "The equity premium: A puzzle." *Journal of Monetary Economics*, 15: 145–161.

Mendoza, Enrique G. 1992. "The effects of macroeconomic shocks in a basic equilibrium f." *Staff Papers (International Monetary Fund)*, 39(4): 855–889.

Obstfeld, Maurice, and Kenneth Rogoff. 2002. "Global Implications of Self-Oriented National Monetary Rules." *The Quarterly Journal of Economics*, 117(2): 503–535.

Ottonello, Pablo, and Diego J Perez. 2019. "The currency composition of sovereign debt." *American Economic Journal: Macroeconomics*, 11(3): 174–208.

Samuelson, Paul A. 1970. "The fundamental approximation theorem of portfolio analysis in terms of means, variances and higher moments." *Review of Economic Studies*, 37: 537–542.

Schmitt-Grohe, Stephanie, and Martín Uribe. 2004. "Optimal fiscal and monetary policy with sticky wages and sticky prices." *Journal of Economic Theory*, , (114): 198–230.

Senay, Ozge, and Alan Sutherland. 2019. "Optimal monetary policy, exchange rate misalignments and incomplete financial markets." *Journal of International Economics*, 117: 196–208.

Sheedy, Kevin D. 2014. "Debt and Incomplete Financial Markets: A Case for Nominal GDP Targeting." *Brookings Papers on Economic Activity*, 2014(1): 301–373.

Siu, Henry E. 2006. "Optimal fiscal and monetary policy with sticky wages and sticky prices." *Review of Economic Dynamics*, 9(4): 683–714.

Tille, Cédric, and Eric van Wincoop. 2010. "International capital flows." *Journal of International Economics*, 80(2): 157–175.

Verstraete, Matthieu, and Lena Suchanek. 2018. "Understanding Monetary Policy and its Effects: Evidence from Canadian Firms Using the Business Outlook Survey."

Woodford, Michael. 2003. *Interest and prices: Foundations of a theory of monetary policy.* Princeton university press.

Appendix (for online publication)

A An (almost) linear-quadratic (ALQ) approximate problem for optimal policy in DSGE models with portfolio choice

Benigno and Woodford (2012) show how to derive approximate linear-quadratic (LQ) problems that yield, as a solution, a correct linear approximation to the optimal policy in a large class of dynamic-stochastic-general-equilibrium (DSGE) models. A notable exception in this class are problems where agents make portfolio decisions.

In this appendix, I derive an approximate problem with (i) an objective that is quadratic in all endogenous variables conditional on the portfolio, (ii) constraints that are linear in all endogenous variables conditional on the portfolio, and (iii) nonlinear in the portfolio. I show that the solution to this problem yields the optimal steady-state portfolio (i.e. the zero-order portfolio) and a linear approximation to the optimal policy for all other endogenous variables that is locally correct if the planner can control the portfolio, e.g. if she chooses it directly (e.g. public debt) or has access to taxes on financial assets to manipulate the portfolio decision of private agents. I also show that, whenever the planner cannot control the portfolio without restrictions, the solution to this problem is generically an infeasible allocation around the steady state. Therefore, the approximate policy that emerges from this "naive" approach is incorrect.

Section A.1 introduces a general class of dynamic optimization problems with forward-looking constraints and portfolio choice in an environment where a planner can control the portfolio. Section A.2 derives an "almost linear-quadratic" (ALQ) approximate problem associated with any problem in this class. Section A.3 presents and proves my main result: that the first-order conditions of the ALQ problem - including the steady-state portfolio as a control - coincide with a perturbation of the first-order conditions (FOC) of the non-linear problem using a perturbation approach that relies on a bifurcation theorem stated in Judd and Guu (2001). Section A.4 shows what fails when the planner cannot perfectly control the portfolio and faces many no-arbitrage conditions as constraints. I show that in this case the "indeterminacy" problem is of a higher dimension. I derive a "generalized" ALQ problem that is valid in this case but requires one to keep track of an additional quadratic constraint. Finally, Section A.5 presents a mapping of the model in the paper to this general framework.

A.1 Set up

Throughout this section I follow Benigno and Woodford (2012) as closely as possible to ease the comparison with a standard problem without portfolio choice. Like them, I consider an abstract discrete-time dynamic optimal policy problem. The only difference is that I explicitly model (i) a portfolio decision and (ii) the number of "agents" $i = \{0, 1, \ldots, I\}$. To define a useful class of problems, I isolate some endogenous variables from the rest: the excess return on an asset $j = 1, \ldots, J$ over a reference asset $j = 0$, rr_{jt+1}, and the position on these assets θ_{ijt} for agents $i \in \{1, \ldots, I\}$.[36] Crucially, I assume that the portfolio by itself has **no direct effect** on utility or the constraints: its only effect is indirect through the transfers $\mathcal{T}_{it+1} = \sum_{j=1}^{J} \theta_{ijt} rr_{jt+1}$ it creates. Otherwise, the portfolio would be determined at the steady state and risk would not play a first-order

[36] The position of the remaining agent is determined by asset-market clearing. Furthermore, the position on the reference asset is also pinned down in a dynamic model by savings' considerations; not risk. Thus, the dimension of the indeterminacy problem each period is $J \times I$.

role around the deterministic steady state.[37] In such a case, I would not need the tools developed in this paper; the results in Benigno and Woodford (2012) would apply directly. Henceforth, for notational convenience I write $\theta_t \equiv \{\theta_{ijt}\}_{i=1,j=1}^{I,J}$, $rr_t \equiv \{rr_{jt}\}_{j=1}^{J}$ and $\mathcal{T}_t \equiv \{\mathcal{T}_{it}\}_{i=1}^{I}$.

The policy authority wishes to determine the evolution of an endogenous state vector $\{y_t, rr_t, \theta_t\}_{t=t_0}^{\infty}$ for $t \geq t_0$ to maximize an objective of the form

$$V_{t_0} \equiv \mathbb{E}_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \pi(y_t; \xi_t), \tag{49}$$

where $0 < \beta < 1$ is a discount factor and ξ_t is a vector of exogenous disturbances. The evolution of the endogenous states must satisfy a system of backward-looking structural relations for all $t \geq t_0$

$$F(y_t, \mathcal{T}_t, \xi_t; y_{t-1}) = 0 \tag{50}$$

$$X_j(y_t, \xi_t; y_{t-1}) = rr_{jt}, \tag{51}$$

where $X_j(\cdot)$ maps endogenous variables and shocks into the realized excess return of asset j relative to the reference asset 0. Compared to Benigno and Woodford (2012), I explicitly wrote the backward-looking constraint (51) to reflect the definition of the realized excess return and included the transfers $\mathcal{T}_t$ as an explicit argument of the F backward looking constraints. Crucially, this is the only place where θ_t appears.[38]

The evolution of the endogenous states must also satisfy a system of forward-looking structural relations

$$\mathbb{E}_t g(y_t, \xi_t; y_{t+1}) = 0 \tag{52}$$

$$\mathbb{E}_t rr_{jt+1} m_0(y_{t+1}, \xi_{t+1}) = 0 \; j = 1, \ldots, J. \tag{53}$$

that must hold for each $t \geq t_0$, given the vector of initial conditions $(y_{t_0-1}, \theta_{t_0-1})$. Compared to Benigno and Woodford (2012), I explicitly wrote the forward-looking constraint (53) to have the interpretation of a "no-arbitrage" condition, involving the product of the realized excess return and a stochastic discount factor m_0. Crucially, there is only one of these constraints per asset; i.e. there are J constraints per time period; not $J \times I$. Essentially, this implies that the planner has $J \times I$ extra degrees of freedom. Intuitively, one can think of the planner as choosing the portfolio of each agent θ_{it} freely while asset prices (an element of y_t) are determined by the constraint (53).

Henceforth, I assume that all functions are locally analytic around the deterministic steady state (defined below) and that $m_0 > 0$ is a positive function.[39] To have a well-defined problem, I also assume the number of backward-looking constraints $n_F + J$ plus the number of forward-looking constraints $n_g + J$ is less or equal than the number $n_y + J$ of endogenous variables other than the portfolio each period. When they are strictly less, such as the model in this paper, then there is at least one dimension along which policy can vary other than the portfolio. When they are equal, the planner can only choose the optimal portfolio, e.g. solving for optimal macro-prudential policy in a real model. A t_0-optimal commitment (the standard Ramsey policy problem) is then the state-contingent evolution $\{y_t, rr_t, \theta_t\}_{t=t_0}^{\infty}$ consistent with equations (50) - (53) for all $t \geq t_0$ that maximizes (49).

[37]In my framework, risk determines the steady-state portfolio, which in turn determines the first-order behavior of the remaining endogenous variables. In this sense, risk matters to first order.

[38]Note that it is without loss of generality that transfers only appear in F, since I can always include a constraint in F that defines an element of y_t to be equal to the transfers.

[39]Essentially, analytic functions are C^{∞} and locally equal to the power series created by their Taylor series expansion. See Judd and Guu (2001) for a formal definition.

Assumption 1. *All functions in the problem* (π,$\{F_k\}_{k=1}^{n_F}$,$\{X_j\}_{j=1}^{J}$,$\{g_k\}_{k=1}^{n_g}$, m_0) *are locally analytic at the deterministic steady state (i.e. for any* θ_t*).* m_0 *is a positive function. The number of backward-looking constraints* $n_F + J$ *plus the number of forward-looking constraints* $n_g + J$ *is equal or less than the number* $n_y + J$ *of endogenous variables other than the portfolio each period.*

Optimal policy from a "timeless" perspective I follow Benigno and Woodford (2012) to obtain a problem with a recursive structure by adding initial precommitments:

$$g(y_{t_0-1}, \xi_{t_0-1}; y_{t_0}) = \bar{g}_{t_0} \tag{54}$$

$$rr_{jt_0} m_0(y_{t_0}, \xi_{t_0}) = \bar{m}_{jt_0} \quad j = 1, \ldots, J. \tag{55}$$

Compared to Benigno and Woodford (2012), there is an additional pre-commitment: agents are not surprised by the value of the realized excess return of the asset.

Let $V(\bar{g}_{t_0}, \bar{m}_{t_0}; y_{t_0-1}, \theta_{t_0-1}, \xi_{t_0}, \xi_{t_0-1})$ be the maximum achievable value of the objective (49) subject to (50) - (53) for all $t \geq t_0$, (54) and (55) where $\bar{m}_{t_0} = \{\bar{m}_{jt_0}\}_{j=1}^{J}$. Then, the infinite-horizon problem is equivalent to maximizing

$$\pi(y_t, \xi_t) + \beta \mathbb{E}_t V(\bar{g}_{t+1}, \bar{m}_{t+1}; y_t, \theta_t, \xi_{t+1}, \xi_t)$$

subject to

$$F(y_t, \mathcal{T}_t, \xi_t; y_{t-1}) = 0 \tag{56}$$

$$X_j(y_t, \xi_t; y_{t-1}) = rr_{jt} \quad j = 1, \ldots, J. \tag{57}$$

$$g(y_{t-1}, \xi_{t-1}; y_t) = \bar{g}_t \tag{58}$$

$$rr_{jt} m_0(y_t, \xi_t) = \bar{m}_{jt} \quad j = 1, \ldots, J. \tag{59}$$

$$\mathbb{E}_t \bar{g}_{t+1} = 0 \tag{60}$$

$$\mathbb{E}_t \bar{m}_{t+1} = 0 \tag{61}$$

As pointed out by Benigno and Woodford (2012), in the presence of forward-looking constraints, one needs suitably chosen initial conditions for the pre-commitments $\{\bar{g}_{t_0}, \bar{m}_{t_0}\}$ to have a deterministic-steady-state solution to this problem.

The solution to the recursive problem yields policy functions:

$$\begin{aligned}
y_t &= y^* \left(\bar{g}_t, \bar{m}_t, y_{t-1}, \theta_{t-1}, \xi_t, \xi_{t-1}\right) \\
rr_t &= rr^* \left(\bar{g}_t, \bar{m}_t, y_{t-1}, \theta_{t-1}, \xi_t, \xi_{t-1}\right) \\
\theta_t &= \theta^* \left(\bar{g}_t, \bar{m}_t, y_{t-1}, \theta_{t-1}, \xi_t, \xi_{t-1}\right) \\
\bar{g}_{t+1} &= \bar{g}^* \left(\xi_{t+1}; \bar{g}_t, \bar{m}_t, y_{t-1}, \theta_{t-1}, \xi_t, \xi_{t-1}\right) \\
\bar{m}_{t+1} &= \bar{m}^* \left(\xi_{t+1}; \bar{g}_t, \bar{m}_t; y_{t-1}, \theta_{t-1}, \xi_t, \xi_{t-1}\right)
\end{aligned}$$

Following Benigno and Woodford (2012), I assume that there is an extended state vector that depends only on the evolution of $(y_t, \theta_t, rr_t, \xi_t)$, defined recursively as

$$\boldsymbol{y}_t = \psi \left(\xi_t, y_t, \theta_t, rr_t, \boldsymbol{y}_{t-1}\right).$$

Plugging in the optimal decisions,

$$\boldsymbol{y}_t = \psi^* \left(\bar{g}_t, \bar{m}_t, y_{t-1}, \theta_{t-1}, \xi_t, \xi_{t-1}, \boldsymbol{y}_{t-1}\right).$$

Next, write the initial pre-commitments (54) and (55) as functions of the current realization of the shock and this extended state vector:

$$\begin{aligned}\bar{g}_{t_0} &= \bar{g}(\xi_{t_0}, \boldsymbol{y_{t_0-1}}) \\ \bar{m}_{t_0} &= \bar{m}_0(\xi_{t_0}, \boldsymbol{y_{t_0-1}}) \quad j = 1, \ldots, J.\end{aligned}$$

for some functions $\bar{g}$, $\bar{m}_0$. These pre-commitments are self-consistent if:

$$\begin{aligned}\bar{g}^*\left(\xi_{t+1}; \bar{g}(\xi_t, \boldsymbol{y}_{t-1}), \bar{m}(\xi_t, \boldsymbol{y}_{t-1}), y_{t-1}, \theta_{t-1}, \xi_t, \xi_{t-1}\right) &= \bar{g}\left(\xi_{t+1}, \psi^*\left(\bar{g}_t, \bar{m}_t, y_{t-1}, \theta_{t-1}, \xi_t, \xi_{t-1}, \boldsymbol{y}_{t-1}\right)\right) \\ \bar{m}^*\left(\xi_{t+1}; \bar{g}(\xi_t, \boldsymbol{y}_{t-1}), \bar{m}(\xi_t, \boldsymbol{y}_{t-1}), y_{t-1}, \theta_{t-1}, \xi_t, \xi_{t-1}\right) &= \bar{m}\left(\xi_{t+1}, \psi^*\left(\bar{g}_t, \bar{m}_t, y_{t-1}, \theta_{t-1}, \xi_t, \xi_{t-1}, \boldsymbol{y}_{t-1}\right)\right)\end{aligned}$$

for all possible values of ξ_{t+1}, ξ_t and $\boldsymbol{y}_{t-1}$. In this case, the initial constraint is of a form that one would optimally commit oneself to satisfy at all subsequent dates. The resulting policy is, in the language of Benigno and Woodford (2012), optimal from a timeless perspective.

A.2 A correct ALQ local approximation

I assume an initial state $(y_{t_0-1}, \theta_{t_0-1})$ and pre-commitments $(\bar{g}_{t_0}, \bar{m}_{t_0})$ such that the optimal policy in the case of zero disturbances is a steady state. The Lagrangian of the nonlinear problem is given by

$$\begin{aligned}\mathcal{L}_{t_0} =& V_{t_0} + \mathbb{E}_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left\{ \sum_{k=1}^{n_F} \lambda_{kt} F_k(y_t, \mathcal{T}_t, \xi_t; y_{t-1}) + \sum_{j=1}^{J} \mu_{jt} \Big(X_j(y_t, \xi_t; y_{t-1}) - rr_{jt} \Big) \right. \\ & \left. + \beta^{-1} \sum_{k=1}^{n_g} \varphi_{kt-1} g_k(y_{t-1}, \xi_{t-1}; y_t) + \beta^{-1} \sum_{j=1}^{J} \eta_{0jt-1} rr_{jt} m_0(y_t, \xi_t) \right\}\end{aligned}$$

where $\lambda_t \equiv \{\lambda_{kt}\}_{k=1}^{n_F}$, $\mu_t \equiv \{\mu_{jt}\}_{j=1}^{J}$, $\varphi_t \equiv \{\varphi_{kt}\}_{k=1}^{n_g}$, and $\eta_{0t} \equiv \{\eta_{0jt}\}_{j=1}^{J}$ are the Lagrange multipliers associated with constraints (50) - (53) respectively, for all $t \geq t_0$, and I use the notation $\beta^{-1}\varphi_{t_0-1}$ and $\beta^{-1}\eta_{0jt_0-1}$ for the pre-commitment constraints (54) and (55). Again, note that the only difference with Benigno and Woodford (2012) is that I explicitly separated the collection of backward-looking and forward-looking constraints that are related to financial assets.

Optimality requires that $\{y_t, rr_t, \theta_t, \lambda_t, \mu_t, \varphi_t, \eta_{0t}\}_{t=t_0}^{\infty}$ satisfy

$$D_y\pi(y_t,\xi_t) + \sum_{k=1}^{n_F}\lambda_{kt}D_yF_k(y_t,\mathcal{T}_t,\xi_t;y_{t-1}) + \beta\mathbb{E}_t\sum_{k=1}^{n_F}\lambda_{kt+1}D_{\check{y}}F_k(y_{t+1},\mathcal{T}_{t+1},\xi_{t+1};y_t)$$
$$+\sum_{j=1}^{J}\mu_{jt}D_yX_j(y_t,\xi_t;y_{t-1}) + \beta\mathbb{E}_t\sum_{j=1}^{J}\mu_{jt+1}D_{\check{y}}X_j(y_{t+1},\xi_{t+1};y_t)$$
$$+\beta^{-1}\sum_{k=1}^{n_g}\varphi_{kt-1}D_{\hat{y}}g(y_{t-1},\xi_{t-1};y_t) + \mathbb{E}_t\sum_{k=1}^{n_g}\varphi_{kt}D_yg(y_t,\xi_t;y_{t+1}) + \beta^{-1}\sum_{j=1}^{J}\eta_{0jt-1}rr_{jt}D_ym_0(y_t,\xi_t) = 0 \tag{62}$$

$$\sum_{k=1}^{n_F}\sum_{i=1}^{I}\theta_{ijt-1}\lambda_{kt}D_{\mathcal{T}_i}F_k(y_t,\mathcal{T}_t,\xi_t;y_{t-1}) - \mu_{jt} + \beta^{-1}\eta_{0jt-1}m_0(y_t,\xi_t) = 0 \tag{63}$$

$$\mathbb{E}_t rr_{jt+1}\left(\sum_{k=1}^{n_F}\lambda_{kt+1}D_{\mathcal{T}_i}F_k(y_{t+1},\mathcal{T}_{t+1},\xi_{t+1};y_t)\right) = 0, \tag{64}$$

which are the FOC with respect to y_t, rr_{jt}, and θ_{ijt}, respectively. I adopt the notation in Benigno and Woodford (2012) such that $D_y, D_{\hat{y}}$ and $D_{\check{y}}$ denote the row vector of partial derivatives of any of the functions with respect to the elements of y_t, y_{t+1} and y_{t-1}, respectively. As usual, I suppose that the vector of exogenous disturbances $\{\xi_t\}$ can be written as

$$\xi_t = \epsilon u_t$$

$\forall t$, where $\{u_t\}$ is a bounded-vector stochastic process and $\epsilon > 0$ is a scalar. I am interested in approximations that become accurate as $\epsilon \to 0$.

Note that, since m_0 is a positive function, an optimal steady state must satisfy:

$$\bar{rr}_j = 0 \tag{65}$$

Thus, at the optimal steady-state $(\bar{y}, \bar{\lambda}, \bar{\mu}, \bar{\varphi})$ must satisfy

$$D_y\pi(\bar{y},0) + \sum_{k=1}^{n_F}\bar{\lambda}_kD_yF_k(\bar{y},0,0;\bar{y}) + \beta\sum_{k=1}^{n_F}\bar{\lambda}_kD_{\check{y}}F_k(\bar{y},0,0;\bar{y})$$
$$+\sum_{j=1}^{J}\bar{\mu}_jD_yX_j(\bar{y},0;\bar{y}) + \beta\sum_{j=1}^{J}\bar{\mu}_jD_{\check{y}}X_j(\bar{y},0;\bar{y})$$
$$+\beta^{-1}\sum_{k=1}^{n_g}\bar{\varphi}_kD_{\hat{y}}g_k(\bar{y},0;\bar{y}) + \sum_{k=1}^{n_g}\bar{\varphi}_kD_yg_k(\bar{y},0;\bar{y}) = 0 \tag{66}$$

$$F(\bar{y},0,0;\bar{y}) = 0 \tag{67}$$

$$X(\bar{y},0;\bar{y}) = 0 \tag{68}$$

$$g(\bar{y},0;\bar{y}) = 0 \tag{69}$$

which is exactly the same as in Benigno and Woodford (2012).[40][41] This reflects the fact that having access to more than one financial asset is irrelevant at the steady state. This also implies that, given that $\bar{rr}_j = 0$, the first-order condition with respect to θ_{ijt} automatically holds (equation 64). Thus, there is the well-known issue that any portfolio solves the problem at the deterministic steady state. Henceforth, I assume that this steady state $(\bar{y}, \bar{\lambda}, \bar{\mu}, \bar{\varphi})$ exists. Given a portfolio $\bar{\theta}_t$, $\bar{\eta}_{0t}$ is then given by:

$$\bar{\eta}_{0jt} = -\beta m_0(\bar{y}, 0)^{-1} \left(\sum_{k=1}^{n_F} \sum_{i=1}^{I} \bar{\theta}_{ijt} \bar{\lambda}_k D_{\mathcal{T}_i} F_k(\bar{y}, 0, 0; \bar{y}) - \bar{\mu}_j \right).$$

A second-order Taylor-series expansion of the objective function π around $(\bar{y}, 0)$ yields

$$\pi(y, \xi) = D_y \pi \cdot \tilde{y} + \frac{1}{2} \tilde{y}' D^2_{yy} \pi \cdot \tilde{y} + \tilde{y}' D^2_{y\xi} \pi \cdot \xi + \text{t.i.p.} + \mathcal{O}(\epsilon^3), \tag{70}$$

where $\tilde{y} \equiv y_t - \bar{y}$ and "t.i.p." refers to terms that are independent of policy. Substituting (70) into (49),

$$V_{t_0} = \mathbb{E}_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left\{ D_y \pi \cdot \tilde{y}_t + \frac{1}{2} \tilde{y}_t' D^2_{yy} \pi \cdot \tilde{y}_t + \tilde{y}_t' D^2_{y\xi} \pi \cdot \xi_t \right\} + \text{t.i.p.} + \mathcal{O}(\epsilon^3). \tag{71}$$

Using a second-order Taylor approximation of each F_k constraint around $(\bar{y}, 0, 0, \{\bar{\theta}_t\}_{t=t_0}^{\infty})$ for an arbitrary $\{\bar{\theta}_t\}_{t=t_0}^{\infty}$,[42] premultiplying by the Lagrange multiplier $\bar{\lambda}_k$ and computing the discounted sum yields

$$\begin{aligned}
\mathbb{E}_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \bar{\lambda}_k F_k(y_t, \mathcal{T}_t, \xi_t; y_{t-1}) = \mathbb{E}_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \bar{\lambda}_k \Bigg\{ & (D_y F_k + \beta D_{\check{y}} F_k) \cdot \tilde{y}_t + \sum_{j=1}^{J} \sum_{i=1}^{I} D_{\mathcal{T}_i} F_k \cdot \left(\bar{\theta}_{ijt-1} rr_{jt} \right) \\
& + \frac{1}{2} \tilde{y}_t' \left(D^2_{yy} F_k + \beta D^2_{\check{y}\check{y}} F_k \right) \cdot \tilde{y}_t + \sum_{j=1}^{J} \sum_{i=1}^{I} \tilde{y}_t' D^2_{y\mathcal{T}_i} F_k \cdot \left(\bar{\theta}_{ijt-1} rr_{jt} \right) \\
& + \tilde{y}_t' D^2_{y\check{y}} F_k \cdot \tilde{y}_{t-1} + \tilde{y}_t' D^2_{y\xi} F_k \cdot \xi_t + \beta \tilde{y}_t' D^2_{\check{y}\xi} F_k \cdot \xi_{t+1} \\
& + \frac{1}{2} \sum_{j=1}^{J} \sum_{j'=1}^{J} \sum_{i=1}^{I} \sum_{i'=1}^{I} \left(\bar{\theta}_{ijt-1} rr_{jt} \right) D^2_{\mathcal{T}_i \mathcal{T}_i'} F_k \cdot \left(\bar{\theta}_{i'j't-1} rr_{j't} \right) \\
& + \sum_{j=1}^{J} \sum_{i=1}^{I} D_{\mathcal{T}_i} F_k \cdot \left(\tilde{\theta}_{ijt-1} rr_{jt} \right) + \sum_{j=1}^{J} \sum_{i=1}^{I} \tilde{y}_{t-1}' D^2_{\check{y}\mathcal{T}_i} F_k \cdot \left(\bar{\theta}_{ijt-1} rr_{jt} \right) \\
& + \sum_{j=1}^{J} \sum_{i=1}^{I} \left(\bar{\theta}_{ijt-1} rr_{jt} \right) D^2_{\mathcal{T}_i \xi} F_k \cdot \xi_t \Bigg\} + \text{t.i.p.} + \mathcal{O}(\epsilon^3).
\end{aligned} \tag{72}$$

where $\tilde{\theta}_t = \theta_t - \bar{\theta}_t$ and I used that realized excess returns at the steady state are zero, i.e. $\bar{rr} = 0$. A first-order approximation of the no-arbitrage constraint (53) yields $\mathbb{E}_{t-1} rr_{jt} = \mathcal{O}(\epsilon^2)\ \forall j$. Noting that $\tilde{\theta}_{it-1}$ and $\tilde{y}_{t-1}$ are predetermined at t and using the law of iterated expectations, (72) simplifies

[40] Recall that constraints $\{X_j\}$ are a special case of a backwards constraint.

[41] I assume an initial pre-commitment $\bar{g}_{t_0}$ near zero in the absence of shocks.

[42] Here, I am abusing notation; $\{\bar{\theta}_t\}_{t=t_0}^{\infty}$ is a full contingent plan. That is, let $s^t = \{\xi_0, \ldots, \xi_t\}$ denote the history until s^t and S^t the set of possible histories at t. I am formally approximating around an arbitrary plan $\{\bar{\theta}(s^t)\}_{s^t \in S^t, t \geq t_0}$.

to[43]

$$\mathbb{E}_{t_0}\sum_{t=t_0}^{\infty}\beta^{t-t_0}\bar{\lambda}_k F_k(y_t,\mathcal{T}_t,\xi_t;y_{t-1}) = \mathbb{E}_{t_0}\sum_{t=t_0}^{\infty}\beta^{t-t_0}\bar{\lambda}_k\Bigg\{ (D_yF_k+\beta D_{\check{y}}F_k)\cdot\tilde{y}_t + \sum_{j=1}^{J}\sum_{i=1}^{I} D_{\mathcal{T}_i}F_k\cdot\left(\bar{\theta}_{ijt-1}rr_{jt}\right)$$

$$+\frac{1}{2}\tilde{y}_t'\left(D^2_{yy}F_k+\beta D^2_{\check{y}\check{y}}F_k\right)\cdot\tilde{y}_t + \sum_{j=1}^{J}\sum_{i=1}^{I}\tilde{y}_t' D^2_{y\mathcal{T}_i}F_k\cdot\left(\bar{\theta}_{ijt-1}rr_{jt}\right)$$

$$+\tilde{y}_t'D^2_{y\check{y}}F_k\cdot\tilde{y}_{t-1}+\tilde{y}_t'D^2_{y\xi}F_k\cdot\xi_t+\beta\tilde{y}_t'D^2_{\check{y}\xi}F_k\cdot\xi_{t+1}$$

$$+\frac{1}{2}\sum_{j=1}^{J}\sum_{j'=1}^{J}\sum_{i=1}^{I}\sum_{i'=1}^{I}\left(\bar{\theta}_{ijt-1}rr_{jt}\right)D^2_{\mathcal{T}_i\mathcal{T}_i'}F_k\cdot\left(\bar{\theta}_{i'j't-1}rr_{j't}\right)$$

$$+\sum_{j=1}^{J}\sum_{i=1}^{I}\left(\bar{\theta}_{ijt-1}rr_{jt}\right)D^2_{\mathcal{T}_i\xi}F_k\cdot\xi_t\Bigg\}+\text{t.i.p.}+\mathcal{O}(\epsilon^3).$$

Note that, compared to Benigno and Woodford (2012), there are additional terms that explicitly consider the transfers implied by the portfolio decision.

I proceed in the same way to approximate each $\{X_j = rr_j\}_{j=1}^J$ constraint,

$$\mathbb{E}_{t_0}\sum_{t=t_0}^{\infty}\beta^{t-t_0}\bar{\mu}_j(X_j(y_t,\xi_t;y_{t-1})-rr_{jt}) = \mathbb{E}_{t_0}\sum_{t=t_0}^{\infty}\beta^{t-t_0}\bar{\mu}_j\Bigg\{ (D_yX_j+\beta D_{\check{y}}X_j)\cdot\tilde{y}_t - rr_{jt}+ \tag{73}$$

$$+\frac{1}{2}\tilde{y}_t'\left(D^2_{yy}X_j+\beta D^2_{\check{y}\check{y}}X_j\right)\cdot\tilde{y}_t+\tilde{y}_t'D^2_{y\check{y}}X_j\cdot\tilde{y}_{t-1}$$

$$+\tilde{y}_t'D^2_{y\xi}X_j\cdot\xi_t+\beta\tilde{y}_t'D^2_{\check{y}\xi}X_j\cdot\xi_{t+1}\Bigg\}+\text{t.i.p.}+\mathcal{O}(\epsilon^3).$$

The approximation of each g_k constraint is identical to Benigno and Woodford (2012),

$$\mathbb{E}_{t_0}\sum_{t=t_0}^{\infty}\beta^{t-t_0-1}\bar{\varphi}_k g_k(y_{t-1},\xi_{t-1};y_t) = \mathbb{E}_{t_0}\sum_{t=t_0}^{\infty}\beta^{t-t_0}\bar{\varphi}_k\Bigg\{\left(D_yg_k+\beta^{-1}D_{\hat{y}}g_k\right)\cdot\tilde{y}_t$$

$$+\frac{1}{2}\tilde{y}_t'\left(D^2_{yy}g_k+\beta^{-1}D^2_{\hat{y}\hat{y}}g_k\right)\cdot\tilde{y}_t+\tilde{y}_t'D^2_{y\xi}g_k\cdot\xi_t$$

$$+\beta^{-1}\tilde{y}_t'D^2_{\hat{y}\xi}g_k\cdot\xi_{t-1}+\beta^{-1}\tilde{y}_t'D^2_{y\hat{y}}g_k\cdot\tilde{y}_{t-1}\Bigg\}+\text{t.i.p.}+\mathcal{O}(\epsilon^3).$$

[43]Since $\bar{rr}=0$, the promise-keeping constraint at $t=t_0$ is, to first order,

$$rr_{jt_0}=\bar{m}_{jt_0}\quad j=1,\ldots,J.$$

This implies that the terms $\tilde{\theta}_{ijt_0-1}rr_{jt_0}$and $\tilde{y}_{t_0-1}'D^2_{\check{y}\mathcal{T}_i}F_k\cdot\left(\bar{\theta}_{ijt_0-1}rr_{jt_0}\right)$ are independent of policy.

Putting everything together, I obtain

$$\begin{aligned}
\beta^{-1}\sum_{k=1}^{n_g}\bar{\varphi}_k\bar{g}_{kt_0} = \mathbb{E}_{t_0}\sum_{t=t_0}^{\infty}\beta^{t-t_0}\Bigg\{ & \Phi_y\tilde{y}_t + \frac{1}{2}\tilde{y}_t'H\tilde{y}_t + \tilde{y}_t'R\tilde{y}_{t-1} + \tilde{y}_t'Z(L)\xi_{t+1} \\
& + \sum_{j=1}^{J}\left(\sum_{k=1}^{n_F}\bar{\lambda}_k\sum_{i=1}^{I}D_{\mathcal{T}_i}F_k\cdot\bar{\theta}_{ijt-1} - \bar{\mu}_j\right)rr_{jt} \\
& + \sum_{k=1}^{n_F}\bar{\lambda}_k\sum_{j=1}^{J}\sum_{i=1}^{I}\tilde{y}_t'D^2_{y\mathcal{T}_i}F_k\cdot\left(\bar{\theta}_{ijt-1}rr_{jt}\right) \\
& + \frac{1}{2}\sum_{k=1}^{n_F}\bar{\lambda}_k\sum_{j=1}^{J}\sum_{j'=1}^{J}\sum_{i=1}^{I}\sum_{i'=1}^{I}\left(\bar{\theta}_{ijt-1}rr_{jt}\right)D^2_{\mathcal{T}_i\mathcal{T}_i'}F_k\cdot\left(\bar{\theta}_{i'j't-1}rr_{j't}\right) \\
& + \sum_{k=1}^{n_F}\bar{\lambda}_k\sum_{j=1}^{J}\sum_{i=1}^{I}\left(\bar{\theta}_{ijt-1}rr_{jt}\right)D^2_{\mathcal{T}_i\xi}F_k\cdot\xi_t\Bigg\} + \text{t.i.p.} + \mathcal{O}(\epsilon^3). \qquad (74)
\end{aligned}$$

where

$$\Phi_y \equiv \sum_{k=1}^{n_F}\bar{\lambda}_k\left(D_yF_k + \beta D_{\check{y}}F_k\right) + \sum_{j=1}^{J}\bar{\mu}_j\left(D_yX_j + \beta D_{\check{y}}X_j\right) + \sum_{k=1}^{n_g}\bar{\varphi}_k\left(D_yg_k + \beta^{-1}D_{\hat{y}}g_k\right)$$

$$H \equiv \sum_{k=1}^{n_F}\bar{\lambda}_k\left(D^2_{yy}F_k + \beta D^2_{\check{y}\check{y}}F_k\right) + \sum_{j=1}^{J}\bar{\mu}_j\left(D^2_{yy}X_j + \beta D^2_{\check{y}\check{y}}X_j\right) + \sum_{k=1}^{n_g}\bar{\varphi}_k\left(D^2_{yy}g_k + \beta^{-1}D^2_{\hat{y}\hat{y}}g_k\right)$$

$$R \equiv \sum_{k=1}^{n_F}\bar{\lambda}_kD^2_{y\check{y}}F_k + \sum_{j=1}^{J}\bar{\mu}_jD^2_{y\check{y}}X_j + \sum_{k=1}^{n_g}\bar{\varphi}_k\beta^{-1}D^2_{\hat{y}y}g_k$$

$$Z(L) \equiv \sum_{k=1}^{n_F}\bar{\lambda}_k\left(\beta D^2_{\check{y}\xi}F_k + D^2_{y\xi}F_k\cdot L\right) + \sum_{j=1}^{J}\bar{\mu}_j\left(\beta D^2_{\check{y}\xi}X_j + D^2_{y\xi}X_j\cdot L\right) + \sum_{k=1}^{n_g}\bar{\varphi}_k\left(D^2_{y\xi}g_k\cdot L + \beta^{-1}D^2_{\hat{y}\xi}g_k\cdot L^2\right).$$

At the steady state,

$$\Phi_y = -D_y\pi$$

Similar to Benigno and Woodford (2012), I can use these relationships to obtain an alternative quadratic approximation to (49),

$$
\begin{aligned}
V_{t_0} = \mathbb{E}_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \Bigg\{ & \frac{1}{2} \tilde{y}_t' Q \cdot \tilde{y}_t + \tilde{y}_t' R \cdot \tilde{y}_{t-1} + 2 \tilde{y}_t B(L) \xi_{t+1} \\
& + \sum_{j=1}^{J} \left(\sum_{k=1}^{n_F} \bar{\lambda}_k \sum_{i=1}^{I} D_{\mathcal{T}_i} F_k \cdot \bar{\theta}_{ijt-1} - \bar{\mu}_j \right) rr_{jt} \\
& + \sum_{k=1}^{n_F} \bar{\lambda}_k \sum_{j=1}^{J} \sum_{i=1}^{I} \tilde{y}_t' D^2_{y\mathcal{T}_i} F_k \cdot \left(\bar{\theta}_{ijt-1} rr_{jt} \right) \\
& + \frac{1}{2} \sum_{k=1}^{n_F} \bar{\lambda}_k \sum_{j=1}^{J} \sum_{j'=1}^{J} \sum_{i=1}^{I} \sum_{i'=1}^{I} \left(\bar{\theta}_{ijt-1} rr_{jt} \right) D^2_{\mathcal{T}_i \mathcal{T}_i'} F_k \cdot \left(\bar{\theta}_{i'j't-1} rr_{j't} \right) \\
& + \sum_{k=1}^{n_F} \bar{\lambda}_k \sum_{j=1}^{J} \sum_{i=1}^{I} \left(\bar{\theta}_{ijt-1} rr_{jt} \right) D^2_{\mathcal{T}_i \xi} F_k \cdot \xi_t \Bigg\} + \text{t.i.p.} + \mathcal{O}(\epsilon^3).
\end{aligned} \tag{75}
$$

where

$$
\begin{aligned}
Q &\equiv D^2_{yy} \pi + H \\
B(L) &\equiv D^2_{y\xi} \pi \cdot L + Z(L)
\end{aligned}
$$

Unlike Benigno and Woodford (2012), however, I am not done: I still have a linear term, dealing with the realized excess return rr_t. Approximating the no-arbitrage constraints (53) to second-order yields

$$
\mathbb{E}_t rr_{jt+1} m_0 + \mathbb{E}_t rr_{jt+1} D_y m_0 \cdot \tilde{y}_{t+1} + \mathbb{E}_t rr_{jt+1} D_\xi m_0 \cdot \xi_{t+1} = \mathcal{O}(\epsilon^3) \tag{76}
$$

where I used that $\bar{rr}_j = 0$. Solving for $\mathbb{E}_t rr_{jt+1}$ and replacing back in (75),[44]

$$
\begin{aligned}
V_{t_0} = \mathbb{E}_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \{ & \frac{1}{2} \tilde{y}_t' Q \cdot \tilde{y}_t + \tilde{y}_t' R \cdot \tilde{y}_{t-1} + \tilde{y}_t B(L) \cdot \xi_{t+1} \\
& + \sum_{j=1}^{J} m_0^{-1} \bar{\mu}_j rr_{jt} \left(D_y m_0 \cdot \tilde{y}_t + D_\xi m_0 \cdot \xi_t \right) \\
& + \sum_{j=1}^{J} \sum_{i=1}^{I} \tilde{y}_t' \mathcal{M}^y_{ij} \cdot \left(\bar{\theta}_{ijt-1} rr_{jt} \right) + \sum_{j=1}^{J} \sum_{i=1}^{I} \xi_t' \mathcal{M}^\xi_{ij} \cdot \left(\bar{\theta}_{ijt-1} rr_{jt} \right) \\
& + \frac{1}{2} \sum_{j=1}^{J} \sum_{j'=1}^{J} \sum_{i=1}^{I} \sum_{i'=1}^{I} \left(\bar{\theta}_{ijt-1} rr_{jt} \right) D_{\mathcal{T}_i \mathcal{T}_i'} F_k \cdot \left(\bar{\theta}_{i'j't-1} rr_{j't} \right) + \text{t.i.p.} + \mathcal{O}(\epsilon^3)
\end{aligned} \tag{77}
$$

[44] When $t = t_0$, I have a similar expression coming from the promise-keeping constraint,

$$
rr_{jt_0} m_0 + rr_{jt_0} D_y m_0 \cdot \tilde{y}_{t_0} + rr_{jt_0} D_\xi m_0 \cdot \xi_{t_0} = \bar{m}_{t_0} + \mathcal{O}(\epsilon^3).
$$

where

$$\mathcal{M}_{ij}^{y} \equiv \sum_{k=1}^{n_F} \bar{\lambda}_k \left(D_{y\mathcal{T}_i}^2 F_k - D_{\mathcal{T}_i} F_k m_0^{-1} D_y m_0 \right)$$
$$\mathcal{M}_{ij}^{\xi} \equiv \sum_{k=1}^{n_F} \bar{\lambda}_k \left(D_{\xi\mathcal{T}_i}^2 F_k - D_{\mathcal{T}_i} F_k m_0^{-1} D_\xi m_0 \right).$$

A linear-approximation of the constraints yields

$$D_y F \cdot \tilde{y}_t + D_{\check{y}} F \cdot \tilde{y}_{t-1} + \sum_{i=1}^{I} \sum_{j=1}^{J} D_{\mathcal{T}_i} F \cdot (\bar{\theta}_{ijt-1} rr_{jt}) + D_\xi F \cdot \xi_t = \mathcal{O}(\epsilon^2) \tag{78}$$

$$D_y X_j \cdot \tilde{y}_t + D_{\check{y}} X_j \cdot \tilde{y}_{t-1} + D_\xi X_j \cdot \xi_t - rr_t = \mathcal{O}(\epsilon^2) \tag{79}$$

$$D_y g \cdot \tilde{y}_t + D_\xi g \cdot \xi_t + D_{\hat{y}} g \cdot \mathbb{E}_t \tilde{y}_{t+1} = \mathcal{O}(\epsilon^2) \tag{80}$$

$$\mathbb{E}_t rr_{t+1} = \mathcal{O}(\epsilon^2). \tag{81}$$

and the additional initial constraints

$$D_y g \cdot \tilde{y}_{t_0-1} + D_\xi g \cdot \xi_{t_0-1} + D_{\hat{y}} g \cdot \tilde{y}_{t_0} = \bar{g}_{t_0} \tag{82}$$

$$rr_{t_0} = \bar{rr}_0 \tag{83}$$

Now, I am ready to define the "almost" linear quadratic problem.

Definition 2. The "almost" linear-quadratic problem is to choose $\{\tilde{y}_t, rr_t, \bar{\theta}_t\}_{t=t_0}^{\infty}$ to maximize (77) subject to (78) - (83).

If it were not for the portfolio, one could maximize this approximate function with respect to a first-order approximation of the constraints. In the language of Benigno and Woodford (2012), the objective is "purely quadratic" in (y_t, rr_t) conditional on $\{\bar{\theta}_t\}_{t=t_0}^{\infty}$ so knowing their first-order behavior would suffice. The issue is, of course, that I do not know what $\{\theta_t\}_{t=t_0}^{\infty}$ is optimal. The "almost" linear-quadratic problem defined above suggests maximizing also with respect to $\{\bar{\theta}_t\}_{t=t_0}^{\infty}$ - a nonlinear problem. Next, I show that such an approach correctly identifies a portfolio that is a local maximizer around the deterministic steady state. As argued above, it is critical for the validity of the approach that the planner has the degrees of freedom to choose portfolios independently. In section A.4, I show that in environments where the portfolio is determined by agents' optimization given the other policy variables, such an approach would violate feasibility in a neighborhood of the steady state. In this case, I show that keeping track of an additional quadratic constraint is unavoidable.

A.3 Equivalence to a linearization of the nonlinear FOC using a bifurcation theorem

In typical models without portfolio choice, one can apply the implicit function theorem to the system of FOC (54) - (64) to determine the existence of functions e.g. $\{y_t(\epsilon)\}_{t=t_0}^{\infty}$ and characterize their first-order behavior with respect to ϵ by differentiation. In the class of problems treated in this appendix, however, the regularity condition fails: the Jacobian is singular. This is easily seen by noting that the no-arbitrage constraint (53) and the planner's FOC with respect to the portfolios

(64) are equivalent to first-order and equal to

$$\mathbb{E}_t rr_{t+1} = \mathcal{O}(\epsilon^2),$$

which implies that the Jacobian drops rank. In other words, the FOCs with respect to the portfolio do not add any restrictions to first-order and the system is underidentified.

I tackle this problem in two steps. First, I consider the system of FOC plus the constraints **except for** the FOC with respect to the portfolio (64) and apply the implicit function theorem to establish the existence of analytic functions $y_t(\epsilon, \{\theta_t\}_{t=t_0}^{\infty})$, $rr_t(\epsilon, \{\theta_t\}_{t=t_0}^{\infty})$, $\lambda_t(\epsilon, \{\theta_t\}_{t=t_0}^{\infty})$, $\varphi_t(\epsilon, \{\theta_t\}_{t=t_0}^{\infty})$, $\mu_t(\epsilon, \{\theta_t\}_{t=t_0}^{\infty})$, and $\eta_t(\epsilon, \{\theta_t\}_{t=t_0}^{\infty})$ around $\epsilon = 0$ and an arbitrary $\{\theta_t\}_{t=t_0}^{\infty}$.[45] Henceforth, I assume this system is exactly identified, i.e. the only equations that drop rank are the FOCs related to the portfolio. This yields:

$$\begin{aligned}\mathbb{E}\left[J(L)\tilde{y}_{t+1}\right] + \mathbb{E}_t\left[B(L)\xi_{t+1}\right] + \sum_{k=1}^{n_F}\mathbb{E}_t\left[M_k^{\lambda}(L)\tilde{\lambda}_{kt+1}\right] + \sum_{j=1}^{J}\mathbb{E}_t\left[M_j^{\mu}(L)\tilde{\mu}_{jt+1}\right] \\ + \sum_{k=1}^{n_g} M_k^{\varphi}(L)\tilde{\varphi}_{kt} + \sum_{i=1}^{I}\sum_{j=1}^{J}\mathcal{M}_{ij}^{y}\cdot\left(\bar{\theta}_{ijt-1}rr_{jt}\right) + \sum_{j=1}^{J} m_0^{-1}(D_y m_0)'\bar{\mu}_j rr_{jt} = \mathcal{O}(\epsilon^2)\end{aligned} \tag{84}$$

$$\begin{aligned}\sum_{i=1}^{I}\bar{\theta}_{ijt-1}\left(\mathcal{M}_{ij}^{y}\right)'\cdot\tilde{y}_t + \sum_{i=1}^{I}\bar{\theta}_{ijt-1}\left(\mathcal{M}_{ij}^{\xi}\right)'\cdot\xi_t + m_0^{-1}\bar{\mu}_j D_y m_0\cdot\tilde{y}_t + m_0^{-1}\bar{\mu}_j D_{\xi} m_0\cdot\xi_t \\ + \sum_{j'=1}^{J}\sum_{k=1}^{n_F}\sum_{i=1}^{I}\sum_{i'=1}^{I}\bar{\theta}_{ijt-1}\bar{\theta}_{i'j't-1}\bar{\lambda}_k D^2_{\mathcal{T}_{i'}\mathcal{T}_i}F_k rr_{j't} + \sum_{i=1}^{I}\sum_{k=1}^{n_F}\bar{\theta}_{ijt-1}D_{\mathcal{T}_i}F_k\cdot\tilde{\lambda}_{kt} + \tilde{\tilde{\eta}}_{0jt-1} - \tilde{\mu}_{jt} = \mathcal{O}(\epsilon^2)\end{aligned} \tag{85}$$

where

$$\begin{aligned}J(L) =& Q\cdot L + R\cdot L^2 + \beta R' \\ M_k^{\lambda}(L) =& \left(\sum_{k=1}^{n_F}(D_y F_k)'\right)\cdot L + \beta\sum_{k=1}^{n_F}(DF_{\check{y}})' \\ M_j^{\mu}(L) =& \left(\sum_{j=1}^{J}(D_y X_j)'\right)\cdot L + \beta\sum_{j=1}^{J}(D_{\check{y}}X_j)' \\ M_k^{\varphi}(L) =& \left(\beta^{-1}\sum_{i=1}^{n_g}(D_{\hat{y}}g_k)'\right)\cdot L + \sum_{k=1}^{n_g}(D_y g_k)' \\ \tilde{\tilde{\eta}}_{0jt-1} =& \beta^{-1}m_0\tilde{\eta}_{0jt-1} + \sum_{i=1}^{I}\sum_{k=1}^{n_F}\bar{\theta}_{ijt-1}\bar{\lambda}_k D^2_{\mathcal{T}_i\check{y}}F\cdot\tilde{y}_{t-1} + \sum_{i=1}^{I}\bar{\lambda}_k D_{\mathcal{T}_i}F_k\cdot\tilde{\theta}_{ijt-1}\end{aligned}$$

where I used that $\bar{rr} = 0$ and $\mathbb{E}_t rr_{t+1} = \mathcal{O}(\epsilon^2)$. Note that $\tilde{\tilde{\eta}}_{t-1}$ is just a translation of the Lagrange multiplier associated with the constraint (53), which will prove convenient later (all the terms in

[45]The derivative, product and sum of analytic functions is analytic. Thus, I can apply the analytic version of the implicit function theorem (stated as Theorem 3 in Judd and Guu (2001)) to establish that the implied implicit functions are also analytic.

the definition are predetermined at t).

This system of equations holds given an arbitrary $\{\bar{\theta}_t\}_{t=t_0}^{\infty}$ but I do not know whether $\{\bar{\theta}_t\}_{t=t_0}^{\infty}$ is a solution to the first-order conditions of the non-linear problem as $\epsilon \to 0$. I solve the problem forward, e.g. I solve for θ_{t_0} taking $\{\theta_t\}_{t>t_0}$ as parameters of the problem. For this, I need to use the set of equations I have not used yet: the FOC with respect to the portfolio given by (64), which can written as

$$H_{ijt_0}\left(\bar{\theta}_{t_0}, \epsilon; \{\bar{\theta}_t\}_{t>t_0}^{\infty}\right) = 0 \tag{86}$$

where I used the result from the implicit function theorem in the first step to define

$$H_{ijt_0}\left(\bar{\theta}_{t_0}, \epsilon; \{\bar{\theta}_t\}_{t>t_0}^{\infty}\right) \equiv \mathbb{E}_t rr_{jt_0+1}\left(\bar{\theta}_{t_0}, \epsilon; \{\bar{\theta}_t\}_{t>t_0}^{\infty}\right) \times \sum_{k=1}^{n_F} \left(\lambda_{kt_0+1}\left(\bar{\theta}_{t_0}, \epsilon; \{\bar{\theta}_t\}_{t>t_0}^{\infty}\right) D_{\mathcal{T}_i} F_k\left(\bar{\theta}_{t_0}, \epsilon; \{\bar{\theta}_t\}_{t>t_0}^{\infty}\right)\right).$$

(I abuse notation in the argument of F_k). Note that, since I assumed that ξ_t was a bounded-vector stochastic process and that all functions are analytic, H_{ijt} is also an analytic function.[46]

I am interested in finding functions $\theta_{ijt_0}(\epsilon; \{\bar{\theta}_t\}_{t>t_0}^{\infty})$ that are valid as $\epsilon \to 0$. That is, I want to find a bifurcation point. To do so, I use a bifurcation theorem stated in Judd and Guu (2001) (Theorem 5), reproduced below for convenience.

Theorem 1. *Suppose* H*:* $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ *is analytic near* $(x_0, 0)$ *and* $H(x, 0) = 0\ \forall x \in \mathbb{R}^n$*. Furthermore, suppose that*

$$D_x H(x_0, 0) = 0_{n \times n}$$
$$D_\epsilon H(x_0, 0) = 0_n$$
$$det(D_{x\epsilon}^2(x_0, 0)) \neq 0.$$

Then, there is an open neighborhood $\mathcal{N}$ *of* $(x_0, 0)$ *and an analytic function* $h(\epsilon) : \mathbb{R} \to \mathbb{R}^n$ *such that* $h(\epsilon) \neq 0$ *for* $\epsilon \neq 0$ *and* $H(h(\epsilon), \epsilon) = 0$ *for* $(h(\epsilon), \epsilon) \in \mathcal{N}$*.*

Consider the optimal portfolio choice at $t = t_0$. Clearly, $D_{\theta_{t_0}} H_{t_0}\left(\bar{\theta}_{t_0}, \epsilon; \{\bar{\theta}_t\}_{t>t_0}^{\infty}\right) = 0$ since $\bar{rr}_j = 0$; i.e. the portfolio is irrelevant at the steady state. Typically, one finds the bifurcation point by looking for x_0 such that $H_\epsilon(x_0, 0) = 0$. Since $\mathbb{E}_t \frac{\partial rr_{jt+1}}{\partial \epsilon} = 0$, however, I have that $D_\epsilon H_{t_0}\left(\bar{\theta}_{t_0}, \epsilon; \{\bar{\theta}_t\}_{t>t_0}^{\infty}\right) = 0\ \forall (\bar{\theta}_{t_0}, 0; \{\bar{\theta}_t\}_{t>t_0}^{\infty})$. Thus, this approach does not work. Indeed, the theorem's regularity condition is not satisfied: I have that $D_{x\epsilon}^2 H = 0\ \forall (\bar{\theta}_{t_0}, 0; \{\bar{\theta}_t\}_{t>t_0}^{\infty})$. To see this, first compute $H_{x\epsilon}$ to obtain

$$\frac{\partial H_{ijt_0}}{\partial \epsilon \partial \theta_{i'j't_0}} = \sum_{k=1}^{n_F} \bar{\lambda}_k D_{\mathcal{T}_i} F_k \left(\mathbb{E}_t \frac{\partial rr_{jt_0+1}}{\partial \epsilon \partial \theta_{i'j't_0}}\right).$$

Next, note that differentiating the no-arbitrage condition (53) with respect to ϵ and θ around $\epsilon = 0$ and $\{\bar{\theta}_t\}_{t=t_0}^{\infty}$ yields:

$$\mathbb{E}_t \frac{rr_{jt_0+1}}{\partial \epsilon \partial \theta_{ij't_0}} = 0,$$

which implies the result. Intuitively, no matter what portfolio agents hold, they are always risk-neutral to first-order. Thus, on average the excess return must be zero - for any portfolio.

To make progress, I use the "divide by epsilon" trick (see e.g. Judd and Guu (2001)) and define

[46] The composition of analytic functions is an analytic function.

a new function $\hat{H}_{ijt_0}$ given by

$$\hat{H}_{ijt_0}\left(\bar{\theta}_{t_0},\epsilon;\{\bar{\theta}_t\}_{t>t_0}^{\infty}\right)=\left\{\begin{matrix}\frac{H_{ijt_0}\left(\bar{\theta}_{t_0},\epsilon;\{\bar{\theta}_t\}_{t>t_0}^{\infty}\right)}{\epsilon} & \text{if } \epsilon\neq 0\\ D_\epsilon H_{ijt_0}\left(\bar{\theta}_{t_0},\epsilon;\{\bar{\theta}_t\}_{t>t_0}^{\infty}\right) & \text{if } \epsilon=0\end{matrix}\right\}.$$

Since $H_{ijt_0}\left(\bar{\theta}_{t_0},0;\{\bar{\theta}_t\}_{t>t_0}^{\infty}\right)=0\ \forall\{\bar{\theta}_t\}_{t=t_0}^{\infty}$, I have that $H_{ijt_0}=\epsilon\hat{H}_{ijt_0}$. Thus, I can rewrite (86) as

$$\hat{H}_{ijt_0}\left(\bar{\theta}_{t_0},\epsilon;\{\bar{\theta}_t\}_{t>t_0}^{\infty}\right)=0.$$

First, note that

$$\frac{\partial\hat{H}_{ijt_0}}{\partial\theta_{ijt_0}}|_{(\{\bar{\theta}_t\}_{t=t_0}^{\infty},\epsilon=0)}=\frac{\partial H_{ijt_0}\left(\bar{\theta}_{t_0},\epsilon;\{\bar{\theta}_t\}_{t>t_0}^{\infty}\right)}{\partial\epsilon\partial\theta_{ijt_0}}|_{(\{\bar{\theta}_t\}_{t=t_0}^{\infty},\epsilon=0)}=0.$$

The condition $\det(D^2_{\theta_{t_0}\epsilon}H)\neq 0$ is a regularity condition that I henceforth assume to hold. Note that this will typically be the case in well-behaved portfolio problems: the elements in the diagonal are non-zero as they reflect how the portfolio varies with the risk-premium, i.e.

$$\frac{\partial\hat{H}_{ijt_0}}{\partial\theta_{ijt_0}\partial\epsilon}|_{(\{\bar{\theta}_t\}_{t=t_0}^{\infty},\epsilon=0)}=\frac{\partial H_{ijt_0}\left(\bar{\theta}_{t_0},\epsilon;\{\bar{\theta}_t\}_{t>t_0}^{\infty}\right)}{\partial\epsilon\partial\epsilon\partial\theta}\neq 0.$$

To find the bifurcation point, thus, I set

$$\frac{\partial\hat{H}_{ijt_0}}{\partial\epsilon}|_{(\{\bar{\theta}_t\}_{t=t_0}^{\infty},\epsilon=0)}=\frac{\partial H_{ijt_0}\left(\bar{\theta}_{t_0},\epsilon;\{\bar{\theta}_t\}_{t>t_0}^{\infty}\right)}{\partial\epsilon\partial\epsilon}=0.$$

Note that this is, essentially, a second order approximation of the planner's portfolio optimality condition. Thus, the procedure is analogous to that of Devereux and Sutherland (2007) who use these second-order approximations to solve for a bifurcation point in the system of equations that describe a competitive equilibrium.

A second-order expansion of H_{ijt_0} yields

$$\begin{aligned}\mathbb{E}_{t_0}rr_{jt_0+1}\sum_{k=1}^{n_F}\bar{\lambda}_k D_{\mathcal{T}_i}F_k+\mathbb{E}_{t_0}rr_{jt_0+1}\sum_{k=1}^{n_F}\bar{\lambda}_k\left(D^2_{\mathcal{T}_i y}F_k\cdot\tilde{y}_{t_0+1}+D^2_{\mathcal{T}_i\xi}F_k\cdot\xi_{t_0+1}+D^2_{\mathcal{T}_i\check{y}}F_k\cdot\tilde{y}_{t_0}\right)\\+\mathbb{E}_{t_0}rr_{jt_0+1}\sum_{j'=1}^{J}\sum_{i'=1}^{I}D^2_{\mathcal{T}_i\mathcal{T}_{i'}}F_k\cdot\bar{\theta}_{i'j't_0}rr_{j't_0+1}+\mathbb{E}_{t_0}rr_{jt_0+1}\sum_{k=1}^{n_F}\tilde{\lambda}_{kt_0+1}D_{\mathcal{T}_i}F=\mathcal{O}(\epsilon^3)\end{aligned}$$

The system of equations (84) - (85) together with the linearized constraints (78) - (81) characterize only the first-order behavior of rr_{jt_0+1}. Crucially, note that $\{\bar{\theta}_s\}_{t>t_0}^{\infty}$ are irrelevant to determine the solution - a consequence of certainty equivalence. Since I have a linear term in rr_{jt_0+1}, I need to use the second-order approximation of the no-arbitrage constraint. Using (76) to replace the linear term and that $\mathbb{E}_{t_0}rr_{jt_0+1}=\mathcal{O}(\epsilon^2)$,

$$\begin{aligned}\mathbb{E}_{t_0}rr_{jt_0+1}\left(\mathcal{M}^y_{hj}\cdot\tilde{y}_{t_0+1}+\mathcal{M}^\xi_{hj}\cdot\xi_{t_0+1}\right)\\+\mathbb{E}_{t_0}rr_{jt_0+1}\sum_{j'=1}^{J}\sum_{i'=1}^{I}D^2_{\mathcal{T}_i\mathcal{T}_{i'}}F_k\cdot\bar{\theta}_{i'j't}rr_{j't_0+1}+\mathbb{E}_{t_0}rr_{jt_0+1}\sum_{k=1}^{n_F}\tilde{\lambda}_{kt_0+1}D_{\mathcal{T}_i}F=\mathcal{O}(\epsilon^3)\end{aligned}\quad(87)$$

Now all terms are purely quadratic. Thus, I can evaluate this expression using only the first-order behavior of rr_{jt_0+1}, $\tilde{y}_{t_0+1}$, and $\tilde{\lambda}_{t_0+1}$, which can be found from (84) - (85) together with the linearized constraints (78) - (81). Crucially, this first-order behavior is independent of $\{\bar{\theta}_t\}_{t>t_0}^{\infty}$ due to certainty equivalence. To see this, simply take time-$t+1$ expectations in the system comprised by (84) - (85) and (78) - (81) and note that the porfolio drops out. Thus, $\{\bar{\theta}_t\}_{t>t_0}^{\infty}$ is irrelevant to determine $\bar{\theta}_{t_0}$. This a system of $J \times I$ equations that I can solve to find the $J \times I$ bifurcation points at $t = t_0$; i.e. solutions $\{\bar{\theta}_t\}_{t=t_0}^{\infty}$ to the nonlinear system of equations that are valid as $\epsilon \to 0$. In a non-stationary environment, I could then use the solution at $t = t_0$ to do the same trick for $t = t_0 + 1$ and so on. Here, the problem is stationary so the solution is the same in every period, i.e. $\bar{\theta}_t = \bar{\theta}_{t_0}\ \forall t$.

Remark 1. Because of the certainty equivalence property, optimal portfolio decisions are separable over time. That is, to zero-order, $\{\bar{\theta}_s\}_{s>t_0}^{\infty}$ is irrelevant for the optimal portfolio decision at t_0, $\bar{\theta}_{t_0}$.

Now I am ready for my main result. Note that: (i) the FOC with respect to $\tilde{y}_t$ yields (84); (ii) the FOC with respect to rr_{jt} yields (85); and, finally, the FOC with respect to $\bar{\theta}_{ijt}$ yields (87). Thus, the first-order conditions of the ALQ problem coincide with a perturbation of the non-linear FOC.

Proposition 17. *The first-order conditions of the ALQ problem (definition 2) coincide with the system of equations (84) - (85), (87) and (78) - (81), which are a perturbation of the first-order conditions of the non-linear problem around the deterministic steady state.*

Remark 2. There are typically many solutions to (87). Thus, keeping track of an approximate welfare function not only allows us to check whether the solution is a local optimum, but also find the *best* local optimum.

A.4 The case with multiple no-arbitrage restrictions

Next, I consider a case with $I \times J$ additional constraints per period,

$$\mathbb{E}_t rr_{jt+1} m_i(y_{t+1}, \xi_{t+1}) = 0 \ j = 1, \ldots, J; i = 1, \ldots, I, \tag{88}$$

where m_i are positive functions. These constraints have the natural intepretation of being no-arbitrage conditions for agent i between assets j and 0 (the reference asset). The new problem is, thus, to maximize the objective (49) subject to (50) - (53) and (88).

Consider the new FOC the non-linear problem, where the new terms are highlighted in red:

$$D_y\pi(y_t,\xi_t)+\sum_{k=1}^{n_F}\lambda_{kt}D_yF_k(y_t,\mathcal{T}_t,\xi_t;y_{t-1})+\beta\mathbb{E}_t\sum_{k=1}^{n_F}\lambda_{kt+1}D_{\check{y}}F_k(y_{t+1},\mathcal{T}_{t+1},\xi_{t+1};y_t)$$
$$+\beta\mathbb{E}_t\sum_{j=1}^{J}\mu_{jt+1}D_{\check{y}}X_j(y_{t+1},\xi_{t+1};y_t)+\sum_{j=1}^{J}\mu_{jt}D_yX_j(y_t,\xi_t;y_{t-1})$$
$$+\beta^{-1}\sum_{k=1}^{n_g}\varphi_{kt-1}D_{\hat{y}}g(y_{t-1},\xi_{t-1};y_t)+\mathbb{E}_t\sum_{k=1}^{n_g}\varphi_{kt}D_yg(y_t,\xi_t;y_{t+1})+\beta^{-1}\sum_{i=0}^{I}\sum_{j=1}^{J}\eta_{ijt-1}rr_{jt}D_ym_i(y_t,\xi_t)=0 \tag{89}$$

$$\sum_{k=1}^{n_F}\sum_{i=1}^{I}\theta_{ijt-1}\lambda_{kt}D_{\mathcal{T}_i}F_k(y_t,\mathcal{T}_t,\xi_t;y_{t-1})-\mu_{jt}+\beta^{-1}\sum_{i=0}^{I}\eta_{ijt-1}m_i(y_t,\xi_t)=0 \tag{90}$$

$$\mathbb{E}_t rr_{jt+1}\left(\sum_{k=1}^{n_F}\lambda_{kt+1}D_{\mathcal{T}_i}F_k(y_{t+1},\mathcal{T}_{t+1},\xi_{t+1};y_t)\right)=0, \tag{91}$$

Like before, a steady state satisfies $\bar{\mu}_j=\bar{rr}_j=0$, and $(\bar{y},\bar{\lambda},\bar{\varphi})$ solve (66) - (69). Crucially, however, a portfolio only pins down a linear combination of the Lagrange multipliers $\{\bar{\eta}_i\}_{i=0}^{I}$:

$$\beta^{-1}\bar{\eta}_{0jt}m_0(\bar{y},0)=-\sum_{i=1}^{I}\bar{\theta}_{ijt}\bar{\lambda}_kD_{\mathcal{T}_i}F_k(\bar{y},0,0;\bar{y})+\bar{\mu}_j-\beta^{-1}\sum_{i=1}^{I}\bar{\eta}_{ijt}m_i(\bar{y},0)$$

In other words, the dimensionality of the "indeterminacy" problem is no longer $J\times I$ per period, but rather $2\times J\times I$. Intuitively, the Lagrange multipliers added in this extension are also indeterminate.

Following the same steps as before, it is straightforward to see that (77) is still a valid expansion of the objective function (49). However, maximizing (77) subject to (78) - (83) no longer yields a portfolio that solves the problem as $\epsilon\to 0$. The reason is that such a portfolio typically violates (88). To see this, note that while the FOC of the ALQ problem are still the same, perturbing the

FOC of the nonlinear problem yields[47]

$$\mathbb{E}\left[J(L)\tilde{y}_{t+1}\right] + \mathbb{E}_t\left[B(L)\xi_{t+1}\right] + \sum_{k=1}^{n_F}\mathbb{E}_t\left[M_k^{\lambda}(L)\tilde{\lambda}_{kt+1}\right] + \sum_{j=1}^{J}\mathbb{E}_t\left[M_j^{\mu}(L)\tilde{\mu}_{jt+1}\right]$$
$$+\sum_{k=1}^{n_g} M_k^{\varphi}(L)\tilde{\varphi}_{kt} + \sum_{i=1}^{I}\sum_{j=1}^{J}\mathcal{M}_{ij}^{y}\cdot\left(\bar{\theta}_{ijt-1}rr_{jt}\right) - \sum_{j=1}^{J} m_0^{-1}(D_y m_0)'\bar{\mu}_j rr_{jt}$$
$$+\beta^{-1}\sum_{i=1}^{I}\sum_{j=1}^{J}\left(D_y m_i - m_0^{-1}m_i D_y m_0\right)\bar{\eta}_{ijt-1}rr_{jt} = \mathcal{O}(\epsilon^2) \tag{92}$$

$$\sum_{i=1}^{I}\bar{\theta}_{ijt-1}\left(\mathcal{M}_{ij}^{y}\right)'\cdot\tilde{y}_t + \sum_{I=1}^{I}\bar{\theta}_{ijt-1}\left(\mathcal{M}_{ij}^{\xi}\right)'\cdot\xi_t + \sum_{i=1}^{I}\sum_{k=1}^{n_F}\bar{\theta}_{ijt-1}D_{\mathcal{T}_i}F_k\cdot\tilde{\lambda}_k$$
$$\sum_{j'=1}^{J}\sum_{k=1}^{n_F}\sum_{i=1}^{I}\sum_{i'=1}^{I}\bar{\theta}_{ijt-1}\bar{\theta}_{i'j't-1}\bar{\lambda}_k D_{\mathcal{T}_i\mathcal{T}_{i'}}F_k rr_{j't} + \tilde{\tilde{\eta}}_{0jt-1} - \tilde{\mu}_{jt}$$
$$+\beta^{-1}\sum_{i=1}^{I}\bar{\eta}_{ijt-1}\left(D_y m_i - m_0^{-1}m_i D_y m_0\right)\cdot\tilde{y}_t + \beta^{-1}\sum_{i=1}^{I}\bar{\eta}_{ijt-1}\left(D_\xi m_i - m_0^{-1}m_i D_\xi m_0\right)\cdot\xi_t = \mathcal{O}(\epsilon^2) \tag{93}$$

where

$$\tilde{\tilde{\eta}}_{0jt-1} \equiv \beta^{-1}\sum_{i=0}^{I} m_i\tilde{\eta}_{ijt-1} + \sum_{i=1}^{I}\tilde{\theta}_{ijt-1}\bar{\lambda}_k D_{\mathcal{T}_i}F_k + \sum_{i=1}^{I}\sum_{k=1}^{n_F}\bar{\theta}_{ijt-1}\bar{\lambda}_k D^2_{\mathcal{T}_i\tilde{y}}F\cdot\tilde{y}_{t-1} + \sum_{i=1}^{I}\tilde{\theta}_{ijt-1}\bar{\lambda}_k D_{\mathcal{T}_i}F_k.$$

The terms in red represent the difference between this sytem of equations and the ones from the ALQ problem. Note that, unless all the Lagrange multipliers of the other no-arbitrage conditions $\{\bar{\eta}_{it}\}_{i=1}^{\infty}$ are zero as $\epsilon \to 0$, the systems will differ. This would be the case if the private agents and the planner made the same decision, e.g. the model in this paper when there is an infinite mass of foreign arbitrageurs.

One can still solve this problem using a perturbation approach to identify bifurcation points of the system of non-linear equations, which now include the multipliers. Using the same arguments as before, one can show that this involves approximating the no arbitrage equations (88) to second order:

$$\mathbb{E}_t rr_{jt+1}m_i + \mathbb{E}_t rr_{jt+1}\left(D_y m_i\cdot\tilde{y}_{t+1} + D_\xi m_i\cdot\xi_{t+1}\right) = \mathcal{O}(\epsilon^3)$$

and then using the agent 0 no-arbitrage condition to eliminate the linear term:

$$\mathbb{E}_t rr_{jt+1}\left(\left(D_y m_i - m_0^{-1}m_i D_y m_0\right)\cdot\tilde{y}_{t+1} + \left(D_y m_i - m_0^{-1}m_i D_\xi m_i\right)\cdot\xi_{t+1}\right) = \mathcal{O}(\epsilon^3). \tag{94}$$

This adds the necessary $J\times I$ equations required to pin down the $J\times I$ Lagrange multipliers $\{\bar{\eta}_i\}_{i=1}^{I}$. Furthermore, note that if one amends the ALQ problem to include (94) as a constraint, one obtains once again the same system of equations.

[47] In this new "first step", the parameters are not only $\{\theta_t\}_{t=t_0}^{\infty}$ and ϵ but also the new Lagrange multipliers $\{\eta_{it}\}_{i=1,t=t_0}^{I,\infty}$.

Proposition 18. *The generalized "almost" linear-quadratic problem (GALQ) is to choose* $\{\tilde{y}_t, rr_t, \bar{\theta}_t\}_{t=t_0}^{\infty}$ *to maximize (77) subject to (78) - (83), and (94). The first-order conditions of the GALQ problem coincide with the system of equations (92) - (93), (87), (78) - (81) and (94), which are a perturbation of the first-order conditions of the non-linear problem around the deterministic steady state.*

Remark 3. This problem is significantly less tractable: It has twice as many degrees of indeterminacy and, as a result, twice as many nonlinear equations. This makes showing properties analytically substantially harder. However, it is fairly easy in a computer and the approach can still be used to check the conditions for local maxima and pick the "best" solution whenever there is more than one.

A.5 Mapping to the model

Consider an infinite-horizon version of the model in section 2 (nested in the general framework of appendix B.4) extended to allow for long home-currency bonds that decay at rate $\delta \in (0,1]$, international interest-rate shocks R^* and a convenience-yield shock Ψ, as in section 5.[48] The controls are $y_t = \{C_{Tt}, E_t, nfa_t, R_t\}$, the portfolio is B, and the shocks are $(Y_{Tt}, Z_t, C_t^*, R_t^*, \Psi_t, R_{t-1}^*, \Psi_{t-1})$. ξ_t is defined as the deviation of these shocks with respect to their steady-state values.[49] The objective function comes from the planner's problem

$$\pi(y_t, \xi_t) = V(C_{Tt}, E_t, Z_t).$$

There is just one F constraint, the budget constraint:

$$F_1(y_t, \mathcal{T}_t, \xi_t; y_{t-1}) = nfa_t + C_{Tt} - \mathcal{T}_t - Y_{Tt} - R_{t-1}^* nfa_{t-1}, \tag{95}$$

where $\mathcal{T}_t = rr_t B_{t-1}$ and $nfa_t = B_t + B_t^*$. The realized excess return of the home-currency bond is

$$X(y_t, y_{t-1}, \xi_t) = \frac{R_{t-1}E_{t-1}}{E_t}\left(\Psi_{t-1} + \delta + (1-\delta)R_t^{-1}\right) - R_{t-1}^*.$$

There are no forward looking constraints g; the only forward-looking constraint is the foreign no-arbitrage equation,

$$\mathbb{E}_t rr_{t+1} \underbrace{U'(C_{t+1}^*)}_{=m_0(y_{t+1}, \xi_{t+1})} = 0.$$

This completes the mapping.

Next, suppose that the foreign agents that trade the home-currency bonds are not large relative to home, i.e. C_t^* cannot be taken as given. Instead, assume that foreigners have an endowment $\{Y_t^*\}$ and they can also borrow at gross rate β^{-1}, as in section 4 and appendix B.4. In this case, the controls are $y_t = \{C_{Tt}, C_t^*, E_t, nfa_t, nfa_t^*, R_t\}$, the portfolio is B, and the shocks ξ_t are $(Y_{Tt}, Z_t, Y_t^*, R_t^*, \Psi_t, R_{t-1}^*, \Psi_{t-1})$. In addition to (95), there is another F constraint:

$$F_2(y_t, \mathcal{T}_t, \xi_t; y_{t-1}) = nfa_t^* + C_{Tt}^* + \mathcal{T}_t - Y_{Tt}^* - R_{t-1}^* nfa_{t-1}^*,$$

[48] The static model is not strictly nested in the abstract infinite horizon model I presented in this section, but it is straightforward to construct an analogous proof for this case or, more generally, a case with a finite horizon.

[49] I stack the lag of R_t^* and Ψ_t inside ξ_t to save notation in the general framework and keep it similar to Benigno and Woodford (2012). Equivalently, one can allow ξ_{t-1} to enter directly the X function. Extending the proof to that case is straightforward.

which already imposes market clearing in the home-currency bond. In addition, there is an additional forward looking constraint, the Euler equation of foreigners:

$$g(y_t, \xi_t; y_{t+1}) = \beta R_t^* U^{*\prime}(C_{t+1}^*) - U'(C_t^*).$$

In section B.4, I also analyze a model with sticky prices à la Calvo. In this case, there is an additional F constraint, which defines the inflation rate, and two forward-looking constraints that capture the firms' optimal pricing decision. This system of three equations simplifies after linearization to the Phillips' curve.

B Proofs and extensions

This appendix is organized into five sections. Sections B.1, B.2, and B.3 prove the results in sections 3, 4 and 5, respectively. Section B.4 presents an infinite-horizon version of the model of section 4 with Calvo pricing and formal results that prove robustness in this extended setup. Section B.5 extends the model of section 4 in two different directions: (i) mark-up shocks and (ii) multiple non-tradable sectors, each with their own nominal rigidity. I formally characterize the conditions that lead to a non-zero approximate tax in this environment.

B.1 Proofs for section 3

B.1.1 Proof of lemma 2

The FOC of (16) with respect to e_s yields

$$e_s - e_{dm,s}(\bar{B}) + \chi \bar{B}^2 (e_s - e_{in,s}(\bar{B})) = 0.$$

Rearranging yields the desired result.

B.1.2 Proof of proposition 2

I start by deriving the optimal portfolio. Let $\mathcal{W}(\bar{B})$ denote the objective in (20). After some algebra, the FOC with respect to $\bar{B}$ yields

$$\frac{\partial \mathcal{W}(\bar{B})}{\partial \bar{B}} = \frac{\chi k_0}{\left(1+\chi \bar{B}^2\right)^2} \{\chi \bar{B}^2 \sigma_{e_{dm}\mathcal{T}_{cm}} + \bar{B}\left(\chi \sigma^2_{\mathcal{T}_{cm}} - \sigma^2_{e_{dm}}\right) - \sigma_{e_{dm}\mathcal{T}_{cm}}\} \tag{96}$$

Since the term inside brackets is a quadratic in $\bar{B}$ there are two solutions to $\frac{\partial \mathcal{W}(B)}{\partial \bar{B}} = 0$. However, the only one of them that is a maximum is

$$\bar{B} = -\left(\frac{\left(\chi \sigma^2_{\mathcal{T}_{cm}} - \sigma^2_{e_{dm}}\right) + \sqrt{\left(\chi \sigma^2_{\mathcal{T}_{cm}} - \sigma^2_{e_{dm}}\right)^2 + 4\chi \left(\sigma_{e_{dm}\mathcal{T}_{cm}}\right)^2}}{2\chi \sigma_{e_{dm}\mathcal{T}_{cm}}}\right), \tag{97}$$

which has the opposite sign as $\sigma_{e_{dm}\mathcal{T}_{cm}}$. To see this, note that in an interior optimum, the second-order condition becomes

$$\frac{\partial^2 \mathcal{W}(\bar{B})}{\partial \bar{B}^2} = \frac{\chi k_0}{\left(1+\chi \bar{B}^2\right)^2} \left\{2\chi \bar{B} \sigma_{e_{dm}\mathcal{T}_{cm}} + \left(\chi \sigma^2_{\mathcal{T}_{cm}} - \sigma^2_{e_{dm}}\right)\right\}. \tag{98}$$

Using the first-order condition (96) holding with equality, this becomes

$$\frac{\partial^2 \mathcal{W}(\bar{B})}{\partial \bar{B}^2} = (\frac{\chi k_0}{1+\chi \bar{B}^2}) \frac{\sigma_{e_{dm}\mathcal{T}_{cm}}}{\bar{B}}.$$

Thus, in an interior optimum the portfolio must be of the opposite sign as $\sigma_{e_{dm}\mathcal{T}_{cm}}$ to be a maximum. This proves the first part of proposition 2.

To see that with perfect correlation one can attain the first-best, one only needs to show that there exists a $\bar{B}$ such that

$$e_{dm,s} = e_{in,s}(\bar{B}) + \mathcal{O}(\epsilon^2) \ \forall s$$

Since shocks are perfectly correlated,

$$e_{dm,s} = k_{dm}\epsilon$$
$$e_{in,s}(\bar{B}) = -\frac{1}{\bar{B}} k_{cm}\epsilon$$

for some constants k_{dm}, k_{cm} where ϵ is the only source of uncertainty in this economy, i.e. $z_s = k_z\epsilon$, $y_s = k_y\epsilon$, $c_s^* = k_{c^*}\epsilon$ for some constants k_z, k_y, k_{c^*}. Thus,

$$\bar{B} = -\frac{k_{cm}}{k_{dm}}$$

attains the first-best by closing both gaps in the welfare function. One can then check that this coincides with (97) in the case of perfect correlation.

B.1.3 Proof of proposition 3

This follows directly from (96). In an interior optimum,

$$\frac{\partial^2 \mathcal{W}(\bar{B})}{\partial \chi \partial \bar{B}} = \frac{\chi k_0}{(1+\chi \bar{B}^2)^2} \bar{B}\{\bar{B}\sigma_{e_{dm}\mathcal{T}_{cm}} + \sigma^2_{\mathcal{T}_{cm}}\}.$$

Note that (97) implies

$$\sigma_{e_{dm}\mathcal{T}_{cm}}\bar{B} + \sigma^2_{\mathcal{T}_{cm}} \geq 0.$$

Thus, $\frac{\partial^2 \mathcal{W}(\bar{B})}{\partial \chi \partial \bar{B}}$ has the same sign as $\bar{B}$ and, by the implicit function theorem, $\bar{B}$ increases with χ if $\bar{B} > 0$ and vice versa.

Using the implicit function theorem on (96) immediately implies that $|\bar{B}|$ increases with $\sigma^2_{\mathcal{T}_{cm}}/\sigma^2_{e_{dm}}$.

Next, note that

$$\frac{\partial \mathcal{W}(\bar{B})}{\partial \bar{B} \partial (\sigma_{e_{dm}\mathcal{T}_{cm}}/\sigma^2_{e_{dm}})} = \frac{\chi k_0 \sigma^2_{e_{dm}}}{(1+\chi \bar{B}^2)^2}(\chi \bar{B}^2 - 1).$$

Using the first-order condition (96) holding with equality, this becomes

$$\frac{\partial \mathcal{W}(\bar{B})}{\partial \bar{B} \partial (\sigma_{e_{dm}\mathcal{T}_{cm}}/\sigma^2_{e_{dm}})} = \frac{\chi k_0 \sigma^2_{e_{dm}}}{(1+\chi \bar{B}^2)^2}\left(-\frac{\bar{B}}{\sigma_{e_{dm}\mathcal{T}_{cm}}}\right)\left(\chi \sigma^2_{\mathcal{T}_{cm}} - \sigma^2_{e_{dm}}\right).$$

Thus, by the implicit function theorem, $\chi \sigma^2_{\mathcal{T}_{cm}} - \sigma^2_{e_{dm}} > 0$ implies that increases in the covariance between both targets (an increase in $\sigma_{e_{dm}\mathcal{T}_{cm}}$ when $\sigma_{e_{dm}\mathcal{T}_{cm}} > 0$ or a decrease in $\sigma_{e_{dm}\mathcal{T}_{cm}}$ when $\sigma_{e_{dm}\mathcal{T}_{cm}} < 0$) implies $|\bar{B}|$ decreases. The converse is true when $\chi \sigma^2_{\mathcal{T}_{cm}} - \sigma^2_{e_{dm}} < 0$.

B.1.4 Proof of lemma 3

The derivative of (21) with respect to ω yields

$$\frac{\partial \sigma_e^2}{\partial \omega} = -2(1-\omega)\sigma_{e_{dm}}^2 + 2\omega\sigma_{e_{in}(\bar{B})}^2 + 2(1-2\omega)\sigma_{e_{dm}e_{in}(\bar{B})}$$

In terms of $\mathcal{T}_{cm}$ and e_{dm},

$$\frac{\partial \sigma_e^2}{\partial \omega} = -2(1-\omega)\sigma_{e_{dm}}^2 + 2\omega\frac{1}{\bar{B}^2}\sigma_{\mathcal{T}_{cm}}^2 - 2\frac{1}{\bar{B}}(1-2\omega)\sigma_{\mathcal{T}_{cm}e_{dm}}.$$

Replacing the optimal $\omega = \frac{\chi\bar{B}^2}{1+\chi\bar{B}^2}$ and rearranging,

$$\frac{\partial \sigma_e^2}{\partial \omega} = \frac{2}{\bar{B}}\frac{1}{1+\chi\bar{B}^2}\left(\chi\bar{B}^2\sigma_{\mathcal{T}_{cm}e_{dm}} + \left(\chi\sigma_{\mathcal{T}_{cm}}^2 - \sigma_{e_{dm}}^2\right)\bar{B} - \sigma_{\mathcal{T}_{cm}e_{dm}}\right).$$

Note that the parenthesis coincides with the term in brackets in (96). Thus, if $\bar{B}$ is an interior optimum, $\frac{\partial \sigma_e^2}{\partial \omega} = 0$.

Next, suppose that the solution is at a corner. Consider first the case $\bar{B} = K$. Since $\bar{B}$ cannot be increased further, optimality of $\bar{B}$ implies that the parenthesis is positive. Thus, $\frac{\partial \sigma_e^2}{\partial \omega} > 0$. Next, consider the case $\bar{B} = -K$. Since $\bar{B}$ cannot be decreased further, optimality of $\bar{B}$ implies that the parenthesis is negative. Thus, $\frac{\partial \sigma_e^2}{\partial \omega} > 0$.

B.1.5 Proof of proposition 5

For this proof, note that

$$\frac{\sigma_e^2}{\sigma_{e_{dm}}^2} = (1-\omega)^2 + \omega^2\frac{1}{\bar{B}^2}\frac{\sigma_{\mathcal{T}_{cm}}^2}{\sigma_{e_{dm}}^2} - 2\omega(1-\omega)\frac{1}{\bar{B}}\frac{\sigma_{e_{dm}\mathcal{T}_{cm}}}{\sigma_{e_{dm}}^2}. \tag{99}$$

Part i

The effect of χ is given by:

$$\frac{d\sigma_e^2}{d\chi} = \frac{\partial \sigma_e^2}{\partial \omega}\frac{\partial \omega}{\partial \chi}$$

The proof of lemma 3 established that $\frac{\partial \sigma_e^2}{\partial \omega} > 0$. Noting that the optimal weight is given by $\omega = \frac{\chi\bar{B}^2}{1+\chi\bar{B}^2}$, it follows that $\frac{\partial \omega}{\partial \chi} > 0$. Thus, $\frac{d\sigma_e^2}{d\chi} > 0$. The effect of $\sigma_{\mathcal{T}_{cm}}^2/\sigma_{e_{dm}}^2$ is given by:

$$\frac{d\sigma_e^2}{d(\sigma_{\mathcal{T}_{cm}}/\sigma_{e_{dm}}^2)} = \omega^2\frac{1}{\bar{B}^2} > 0.$$

Part ii The effect of χ is given by

$$\frac{d\sigma_e^2}{d\chi} = \underbrace{\frac{\partial\sigma_e^2}{\partial\omega}}_{=0}\left(\frac{\partial\omega}{\partial\chi} + \frac{\partial\omega}{\partial\bar{B}}\frac{d\bar{B}}{d\chi}\right) + \underbrace{\left(-\omega^2\frac{2}{\bar{B}^2}\frac{\sigma^2_{\mathcal{T}cm}}{\sigma^2_{e_{dm}}} + 2\omega(1-\omega)\underbrace{\frac{1}{\bar{B}}\frac{\sigma_{e_{dm}\mathcal{T}cm}}{\sigma^2_{e_{dm}}}}_{-}\right)}_{-}\underbrace{\frac{1}{\bar{B}}\frac{d\bar{B}}{d\chi}}_{+} < 0.$$

The first effect is the composition effect, which by lemma 3 is zero. The second effect is the change in volatility coming from the insurance target when $\bar{B}$ changes. This term is unambiguously negative. Thus, the total effect is negative.

The effect of $\sigma^2_{\mathcal{T}cm}/\sigma^2_{e_{dm}}$ is given by

$$\begin{aligned}\frac{d\sigma_e^2}{d\sigma^2_{\mathcal{T}cm}/\sigma^2_{e_{dm}}} &= \underbrace{\frac{\partial\sigma_e^2}{\partial\omega}}_{=0}\left(\frac{\partial\omega}{\partial\sigma^2_{\mathcal{T}cm}/\sigma^2_{e_{dm}}} + \frac{\partial\omega}{\partial\bar{B}}\frac{d\bar{B}}{d\sigma^2_{\mathcal{T}cm}/\sigma^2_{e_{dm}}}\right) \\ &+ \left(-\omega^2\frac{2}{\bar{B}^3}\frac{\sigma^2_{\mathcal{T}cm}}{\sigma^2_{e_{dm}}} + 2\omega(1-\omega)\frac{1}{\bar{B}^2}\frac{\sigma_{e_{dm}\mathcal{T}cm}}{\sigma^2_{e_{dm}}}\right)\frac{d\bar{B}}{d\sigma^2_{\mathcal{T}cm}/\sigma^2_{e_{dm}}} + \omega^2\frac{1}{\bar{B}^2}\end{aligned}$$

In this case, there is an additional direct effect given by the last term. After some algebra, using (96) and the optimal ω, one can show that:

$$\begin{aligned}&\left(-\omega^2\frac{2}{\bar{B}^3}\frac{\sigma^2_{\mathcal{T}cm}}{\sigma^2_{e_{dm}}} + 2\omega(1-\omega)\frac{1}{\bar{B}^2}\frac{\sigma_{e_{dm}\mathcal{T}cm}}{\sigma^2_{e_{dm}}}\right)\frac{d\bar{B}}{d\sigma^2_{\mathcal{T}cm}/\sigma^2_{e_{dm}}} + \omega^2\frac{1}{\bar{B}^2} = \\ &\chi^2\bar{B}^2\left(1+\chi\frac{\sigma^2_{\mathcal{T}cm}}{\sigma^2_{e_{dm}}}\right)\left(1+\chi\bar{B}^2\right)^{-2}\left(2\chi\bar{B}\frac{\sigma_{e_{dm}\mathcal{T}cm}}{\sigma^2_{e_{dm}}} + \chi\frac{\sigma^2_{\mathcal{T}cm}}{\sigma^2_{e_{dm}}} - 1\right)^{-1} < 0,\end{aligned}$$

where the last inequality follows from the fact that the second-order condition (98) implies that the last term in parenthesis is negative. Thus, the effect of the insurance target always dominates the direct effect on volatility. It follows that volatility decreases when $\sigma^2_{\mathcal{T}cm}/\sigma^2_{e_{dm}}$ increases.

B.1.6 Proof of proposition 6

In the main text. This is also a special case of proposition 12 (see section B.2.10 for a proof).

B.1.7 Proof of lemma 4

This is immediate from the approximate objective (28).

B.1.8 Proof of proposition 7

Replacing the optimal exchange rate into (28) yields

$$\mathcal{W} = -\frac{1}{2}\frac{\chi k_0}{1+\chi f(\bar{B})^2}\left(\underbrace{f(\bar{B})^2\sigma^2_{e_{dm}(0)}}_{\text{demand-management}} + \underbrace{\sigma^2_{\mathcal{T}cm}}_{\text{insurance}} - \underbrace{2f(\bar{B})\sigma_{\mathcal{T}cm\,e_{dm}(0)}}_{\text{align targets}}\right) + \text{t.i.p.} + \mathcal{O}(\epsilon^3) \qquad (100)$$

Since (100) is identical to (20) with $f(\bar{B}) = -\bar{B}$ and $\sigma^2_{e_{dm}(0)}$ instead of $\sigma^2_{e_{dm}}$, it is immediate that propositions 3, and 4 hold with $f(\bar{B}) = -\frac{\bar{B}}{1-\mu\bar{B}}$ instead of $\bar{B}$ and $e_{dm}(0)$ instead of e_{dm}. Proposition 2 holds in terms of $f(\bar{B})$ with the opposite sign, i.e. $f(\bar{B})$ has the same sign as $\sigma_{\mathcal{T}_{cm}e_{dm}(0)}$.

Rewriting the optimal exchange rate rule (29) in terms of rr_{fs},

$$rr_{fs} = (1-\omega)e_{dm,s}(0) + \frac{1}{f(\bar{B})}\omega\mathcal{T}_{cm,s} + \mathcal{O}(\epsilon^2). \tag{101}$$

Following steps analogous to the proof of lemma 3,

$$\frac{\partial\sigma^2_e}{\partial\omega} = \frac{2}{f(\bar{B})}\frac{1}{1+\chi f(\bar{B})^2}\left(-\chi f(\bar{B})^2\sigma_{\mathcal{T}_{cm}e_{dm}(0)} + \left(\chi\sigma^2_{\mathcal{T}_{cm}} - \sigma^2_{e_{dm}(0)}\right)f(\bar{B}) + \sigma_{\mathcal{T}_{cm}e_{dm}(0)}\right).$$

The term inside the parenthesis has the same sign as the FOC with respect to $f(\bar{B})$. Suppose $\bar{B} = -K$. Then, $f(K) = K/(1-\mu K) > 0$. Optimality of $f(\bar{B})$ implies that the term in parenthesis is positive and thus the whole term is positive. Suppose $\bar{B} = K$ and $1-\mu K > 0$. Then, $f(K) = -K/(1-\mu K) < 0$. Optimality of $f(\bar{B})$ implies that the term in parenthesis is negative and thus the whole term is positive. To see why the condition $1-\mu\bar{B} < 0$ is required, suppose $\bar{B} = K$ and $1-\mu K < 0$. Then, $f(K) = -K/(1-\mu K) > 0$. Note that, in this region, the planner cannot increase $\bar{B}$ which implies it cannot **decrease** $f(\bar{B})$. Thus, optimality of $f(\bar{B})$ implies that the term in parenthesis is negative and thus the whole term is negative.

The proof of part (ii) of proposition 5 is identical with $\sigma^2_{rr_f}/\sigma^2_{e_{dm}(0)}$ instead of $\sigma^2_e/\sigma^2_{e_{dm}}$.

B.1.9 Proof of proposition 8

Note that

$$e_s = \left(\frac{1}{1-\mu\bar{B}}\right)rr_{fs}.$$

Thus, for $x \in \{\chi, \frac{\sigma^2_{\mathcal{T}_{cm}}}{\sigma^2_{e_{dm}(0)}}\}$,

$$\frac{\partial\sigma^2_e}{\partial x} = \frac{1}{(1-\mu\bar{B})^2}\frac{\partial\sigma^2_{rr_f}}{\partial x} + \frac{2\mu}{1-\mu\bar{B}}\sigma^2_e\frac{\partial\bar{B}}{\partial x}.$$

The result in part (i) assumes $|\bar{B}| = K$ so $\frac{\partial\bar{B}}{\partial x} = 0$. Thus, $\text{sign}(\frac{\partial\sigma^2_e}{\partial x}) = \text{sign}(\frac{\partial\sigma^2_{rr_{fs}}}{\partial x})$. Proposition 7 then implies the result. For part (ii), note that when $\bar{B} < 0$, a larger importance of insurance implies both $\frac{\partial\sigma^2_{rr_f}}{\partial x} < 0$ and $\frac{\partial\bar{B}}{\partial x} < 0$ (since $f(\bar{B})$ increases). Since $1-\mu\bar{B} > 0$, $\frac{\partial\sigma^2_e}{\partial x} < 0$. By contrast, when $\bar{B} > 0$ and $1-\mu\bar{B} > 0$, the second term in the parenthesis is positive so the overall sign is ambiguous.

B.1.10 Proof of proposition 9

This is a special case of proposition 12 (see section B.2.10 for a proof).

B.2 Static model: General framework

I present the general static model in detail and prove all the results in section 4. Section B.2.1 describes the set up and characterizes the competitive equilibrium. Section B.2.2 presents the planning problem. Section B.2.3 describes the deterministic steady state. Section B.2.4 derives a second-order approximation of home's flow utility. Section B.2.5 presents and proves lemma 7, which approximates the objective function of the planning problem as a function of three loss terms: lack of insurance, output gaps, and price dispersion. Section B.2.6 presents and proves lemma 8, which writes the objective function in terms of the deviation of exchange rates with respect to two exchange-rate targets, as in the baseline model of section 2. Sections B.2.7, B.2.8, B.2.9, and B.2.10 prove proposition 10, lemma 5 and propositions 11 and 12, respectively.

B.2.1 Set up and competitive equilibrium

Consumers Consumers solve

$$\max_{\{C_{Ts},C_{Ns},L_s\}_s,\{\Theta_j\}_j} \sum_s \pi_s U(C_{Ts}, C_{Ns}, L_s; \xi_s) \tag{102}$$

subject to

$$\sum_j (1+\tau_j)\Theta_j + \Theta_0 = T_0 \tag{103}$$

$$E_s C_{Ts} + P_{Ns} C_{Ns} = E_s Y_T(\xi_s) + W_s L_s + \int_0^1 \Pi_{Ns}(i)di \tag{104}$$
$$+ \sum_{j>0} R_j E_s \tilde{X}_j(\mathcal{Y}_s, \xi_s)\Theta_j + E_s\Theta_0 + T_s.$$

where $\Pi_{Ns}(i)$ are profits from firm i, and

$$\mathcal{Y}_s = \left\{ C_{Ts}, C_{Ns}, L_s, E_s^{-1}P_{Ns}, E_s^{-1}W_s, E_s^{-1}\int_0^1 \Pi_{Ns}(i)di, E_s \right\}$$

are the aggregate ex-post endogenous variables of the model in foreign-currency (taken as given by the representative agent). Optimization over labor and tradable and nontradable consumption yields

$$\frac{\partial U}{\partial C_{Ns}}(s)/\frac{\partial U}{\partial C_{Ts}}(s) = \frac{P_{Ns}}{E_s} \tag{105}$$

$$(-\frac{\partial U}{\partial L_s}(s))/\frac{\partial U}{\partial C_{Ts}}(s) = \frac{W_s}{E_s} \tag{106}$$

Optimization across varieties gives rise to the standard CES demand,

$$C_{Ns}(i) = \left(\frac{P_{Ns}(i)}{P_{Ns}}\right)^{-\eta} C_{Ns}, \tag{107}$$

where P_{Ns} is the ideal price index of nontradable goods,

$$P_{Ns} = \left(\int_0^1 P_{Ns}(i)^{1-\eta}di\right)^{\frac{1}{1-\eta}}. \tag{108}$$

Asset optimization yields a no-arbitrage condition,

$$\sum_s \pi_s \left((1+\tau_j)^{-1}R_j\tilde{X}_j(\mathcal{Y}_s;\xi_s) - 1\right)\frac{\partial U}{\partial C_{Ts}}(s) = 0. \tag{109}$$

Foreigners Foreigner optimization, together with the asset market clearing condition, yields

$$\sum_s \pi_s \left[(R_j \tilde{X}_j(\mathcal{Y}_s;\xi_s) - 1) \frac{dU^*\left(Y^*(\xi_s) - \frac{1}{m}\sum_j (R_j \tilde{X}_j(\mathcal{Y}_s;\xi_s) - 1)\Theta_j\right)}{dC^*} \right] = 0 \tag{110}$$

Intermediate good producers Firms have access to a neoclassical technology

$$C_{Ns}(i) = F(L_s(i);\xi_s). \tag{111}$$

Note that technology is identical across firms and there are no idiosyncratic technology shocks. A set ϕ of firms $i \in \{\text{fix}\}$ cannot reset their price:

$$P_{Ns}(i) = 1 \text{ for } i \in \{\text{fix}\}. \tag{112}$$

A set of firms $1-\phi$ $i \in \{\text{flex}\}$ can reset their price. I assume that there is a labor subsidy τ_L that offsets these firms' desired mark up, i.e. $1 - \tau_L = \frac{\eta - 1}{\eta}$. Optimality gives rise to the condition,

$$P_{Ns}(i) = \frac{1}{\frac{\partial F}{\partial L}(s)} W_s \text{ for } i \in \{\text{flex}\}. \tag{113}$$

Taxes The central government rebates the proceeds of the financial taxes $\{\tau_j\}$ lump-sum at $t = 0$ and the cost of the labor subsidy τ_L lump sum at $t = 1$:

$$T_0 = \sum_j \tau_j \Theta_j$$
$$T_s = -\tau_L W_s L_s.$$

Goods and labor market clearing Solving for Θ_0 using (103), replacing Θ_0, firms' profits and the lump sum taxes into (104), and using nontradable market clearing yields the country's budget constraint:

$$C_{Ts} = Y_T(\xi_s) + \sum_j (R_j \tilde{X}_j(\mathcal{Y}_s, \xi_s) - 1)\Theta_j. \tag{114}$$

The market clearing condition for labor is given by

$$L_s = \int_0^1 L_s(i) di. \tag{115}$$

Competitive equilibrium Next, I formally define a competitive equilibrium in this economy.

Definition 3. Given a Central Bank policy ($\{E_s\}_s$,$\{\tau_j\}_j$), an allocation ($\{C_{Ts}\}_s$,$\{C_{Ns}\}_s$,$\{L_s\}_s$, $\{C_{Ns}(i)\}_{i,s}$,$\{L_s(i)\}_{i,s}$,$\{\Theta_j\}_j$) together with prices ($\{P_{Ns}\}_s$,$\{W_s\}$,$\{P_{Ns}(i)\}_{i,s}$,$\{R_j\}_j$) is a ***competitive equilibrium*** if and only if they solve (105)−(115).

B.2.2 Planning problem

As in Farhi and Werning (2016), I define the following indirect utility function:

$$V(C_{Ts}, E_s; \xi_s) = \max_{\{C_{Ns}, L_s, P_{Ns}, W_s\}_s, \{C_{Ns}(i), L_s(i), P_{Ns}(i)\}_{i,s}} u(C_{Ts}, C_{Ns}, \int_0^1 L_s(i)di; \xi_s) \tag{116}$$
$$\text{subject to}(105) - (108), (111) - (113), (115).$$

Since $\{\tau_j\}_j$ is a policy variable, it can always be chosen to make (109) hold.[50]

Similarly, using the solution to this problem one can write a "reduced-form" return of financial assets:

$$X_j(C_{Ts}, E_s; \xi_s) = \tilde{X}_j(\mathcal{Y}^*(C_{Ts}, E_s; \xi_s); \xi_s)$$

where $\mathcal{Y}^*$ are the endogenous variables, e.g. nontradable consumption, as a function of the two remaining endogenous variables (C_{Ts}, E_s) and shocks ξ_s.

Using these two objects, I can write the problem in a similar way to the one in the main text.

Problem 3. The planner's problem is to choose $\{C_{Ts}, C_s^*, E_s, \{\Theta_j\}_j\}$ to maximize

$$\mathcal{W} = \sum_s \pi_s \left\{ V(C_{Ts}, E_s; \xi_s) + m\bar{\lambda} U^*(C_s^*) \right\}$$

subject to

$$Y_T(\xi_s) + \sum_j (R_j X_j(C_{Ts}, E_s; \xi_s) - 1)\Theta_j = C_{Ts}$$

$$\sum_s \pi_s \left[(R_j X_j(C_{Ts}, E_s; \xi_s) - 1) \frac{dU^*\left(Y^*(\xi_s) - \frac{1}{m}\sum_j (R_j X_j(C_{Ts}, E_s; \xi_s) - 1)\Theta_j\right)}{dC^*} \right] = 0.$$

B.2.3 Steady state

Suppose there is no risk. The first best involves no inefficient fluctuations of consumption

$$C_{Ts} = \bar{C}_T = \bar{Y}_T,$$

all intermediate-input firms producing the same amount since they are all ex ante identical

$$C_{Ns}(i) = C_{Ns} = \bar{C}_N$$
$$\bar{C}_N = F(\bar{L}; 0),$$

and the equalization of marginal rates of substitution and transformation between non-tradable goods and labor

$$\frac{\partial U}{\partial C_N} \frac{\partial F}{\partial L} = -\frac{\partial U}{\partial L},$$

which determines the steady-state $\bar{L}$.

[50] This system of equations typically has a unique solution so there is no real maximization involved. This is the case, for example, in the baseline model where L_s and C_{Ns} are determined by (10) and (11).

Since monetary policy does not play an insurance role at the steady state, it can attain this allocation by ensuring that flexible-price firms would like to set their prices in home currency equal to the fixed-price firms, i.e.

$$\bar{P}_N = \bar{P}_N(i) = 1 \ \forall i.$$

Then,

$$\bar{E} = \frac{\frac{\partial U}{\partial C_T}}{\frac{\partial U}{\partial C_N}} \bar{P}_N$$

$$\bar{W} = \frac{\partial F}{\partial L} \bar{P}_N$$

ensure that (105), (106), and (113) hold. Finally, $\bar{R}_j \bar{E}^{-1} = 1 \ \forall j$. As usual, $\{\bar{\Theta}_j\}_j$ is indeterminate at the deterministic steady state.

B.2.4 Second-order approximation to home's flow utility

In this section, I derive a second-order approximation to the general flow utility function (102).

Lemma 6. *Around the deterministic steady state, home's flow utility is approximately given by*

$$U(C_{Ts}, C_{Ns}, L_s; \xi_s) = \frac{\partial U}{\partial C_T} \bar{C}_T \left(c_{Ts} + \frac{1}{2} c_{Ts}^2 \right) - \frac{1}{2} A_{pp} \left(\int_0^1 (p_{Ns}(i) - p_{Ns})^2 di \right) + \frac{1}{2} \frac{\partial^2 U}{\partial C_T^2} \bar{C}_T^2 c_{Ts}^2 \tag{117}$$

$$+ \bar{C}_T c_{Ts} A_{c\xi} \cdot \xi_s + A_{cl} \bar{C}_T c_{Ts} \bar{L} l_s - \frac{1}{2} A_{ll} \bar{L}^2 l_s^2 + \bar{L} l_s A_{l\xi} \cdot \xi_s + t.i.p. + \mathcal{O}(\epsilon^3)$$

where

$$A_{pp} = \frac{\partial U}{\partial C_N} \bar{C}_N \left(1 - \frac{\bar{C}_N \frac{\partial^2 F}{\partial L^2} \eta}{(\frac{\partial F}{\partial L})^2} \right) \eta > 0 \tag{118}$$

$$A_{cl} = (\frac{\partial^2 U}{\partial C_T \partial C_N}) \frac{\partial F}{\partial L} + \frac{\partial^2 U}{\partial C_T \partial L} \tag{119}$$

$$A_{c\xi} = \frac{\partial^2 U}{\partial C_T \partial C_N} D_\xi F + D^2_{c_T \xi} U \tag{120}$$

$$A_{ll} = -\left(\frac{\partial U}{\partial C_N} \frac{\partial^2 F}{\partial L^2} + \frac{\partial^2 U}{\partial C_N^2} (\frac{\partial F}{\partial L})^2 + 2 (\frac{\partial^2 U}{\partial C_N \partial L}) \frac{\partial F}{\partial L} + \frac{\partial^2 U}{\partial L^2} \right) > 0 \tag{121}$$

$$A_{l\xi} = \frac{\partial U}{\partial C_N} D^2_{L\xi} F + \frac{\partial^2 U}{\partial C_N^2} \frac{\partial F}{\partial L} D_\xi F + (\frac{\partial^2 U}{\partial C_N \partial L}) D_\xi F + \frac{\partial F}{\partial L} D^2_{C_N \xi} U + D^2_{L\xi} U. \tag{122}$$

I use the following notation: $D_x G = \{\frac{\partial G}{\partial x_i}\}_i$ denotes the gradient of function G with respect to x (a row vector), and $D^2_{xy} G$ is the Hessian of G with respect to x and y, i.e. a matrix where element (i, j) denotes $\frac{\partial^2 G}{\partial x_i \partial y_j}$. Note that, in the expressions above, $D^2_{C_N \xi}$ and $D^2_{L\xi}$ are row vectors since C_N and L are scalars. Thus, $A_{c\xi}, A_{l\xi} \in \mathbb{R}^{1 \times K}$.

A second-order approximation of $U(C_{Ts}, C_{Ns}, L_s; \xi_s)$ yields

$$
\begin{aligned}
U(C_{Ts}, C_{Ns}, L_s; \xi_s) =& \frac{\partial U}{\partial C_T}\bar{C}_T c_{Ts} + \frac{\partial U}{\partial C_N}\bar{C}_N c_{Ns} + \frac{\partial U}{\partial L}\bar{L} l_s + \frac{1}{2}(\frac{\partial U}{\partial C_T} + \frac{\partial^2 U}{\partial C_T^2}\bar{C}_T)\bar{C}_T c_{Ts}^2 \\
&+ (\frac{\partial^2 U}{\partial C_T \partial C_N})\bar{C}_T\bar{C}_N c_{Ts} c_{Ns} + (\frac{\partial^2 U}{\partial C_T \partial L})\bar{C}_T\bar{L} c_{Ts} l_s + \bar{C}_T c_{Ts} D^2_{C_T\xi} U \cdot \xi_s \\
&+ \frac{1}{2}(\frac{\partial U}{\partial C_N} + \frac{\partial^2 U}{\partial C_N^2}\bar{C}_N)\bar{C}_N c_{Ns}^2 + (\frac{\partial^2 U}{\partial C_N \partial L})\bar{C}_N\bar{L} c_{Ns} l_s + \bar{C}_N c_{Ns} D^2_{C_N\xi} U \cdot \xi_s \\
&+ \frac{1}{2}(\frac{\partial U}{\partial L} + \frac{\partial^2 U}{\partial L^2}\bar{L})\bar{L} l_s^2 + \bar{L} l_s D^2_{L\xi} U \cdot \xi_s + \text{t.i.p.} + \mathcal{O}(\epsilon^3).
\end{aligned} \tag{123}
$$

Let Δ_s be defined as

$$
\Delta_s = \frac{F(L_s; \xi_s)}{\left(\int_0^1 \left(F(L_s(i); \xi_s)\right)^{\frac{\eta-1}{\eta}}\right)^{\frac{\eta}{\eta-1}}},
$$

which is a measure of output dispersion across variety producers, and rewrite the definition of the composite nontradable good as follows,

$$
C_{Ns} = \Delta_s^{-1} F(L_s; \xi_s).
$$

A second-order approximation yields

$$
\bar{C}_N\left(c_{Ns} + \frac{1}{2}c_{Ns}^2\right) = -\Delta_s\bar{C}_N + \frac{\partial F}{\partial L}\bar{L}(l_s + \frac{1}{2}l_s^2) + \frac{1}{2}\frac{\partial^2 F}{\partial L^2}\bar{L}^2 l_s^2 + \bar{L} l_s D_{L\xi} F \cdot \xi_s + \text{t.i.p.} + \mathcal{O}(\epsilon^3) \tag{124}
$$

where I used the fact that there is no dispersion to first order, i.e. $\Delta_s = \mathcal{O}(\epsilon^2)$. A second-order approximation of Δ_s yields

$$
\Delta_s = \frac{1}{2}\left(1 - \frac{\bar{C}_N \frac{\partial^2 F}{\partial L^2}\eta}{(\frac{\partial F}{\partial L})^2}\right)\eta\left(\int_0^1 (p_{Ns}(i) - p_{Ns})^2 di\right) + \mathcal{O}(\epsilon^3).
$$

Replacing back in (124),

$$
\begin{aligned}
\bar{C}_N\left(c_{Ns} + \frac{1}{2}c_{Ns}^2\right) =& -\frac{1}{2}\left(1 - \frac{\bar{C}_N \frac{\partial^2 F}{\partial L^2}\eta}{(\frac{\partial F}{\partial L})^2}\right)\eta\bar{C}_N\left(\int_0^1 (p_{Ns}(i) - p_{Ns})^2 di\right) \\
&+ \frac{\partial F}{\partial L}\bar{L}(l_s + \frac{1}{2}l_s^2) + \frac{1}{2}\frac{\partial^2 F}{\partial L^2}\bar{L}^2 l_s^2 + \bar{L} l_s D^2_{L\xi} F \cdot \xi_{ks} + \text{t.i.p.} + \mathcal{O}(\epsilon^3)
\end{aligned} \tag{125}
$$

Replacing (125) into (123), discarding higher-order terms and simplifying yields the desired expression.

B.2.5 Approximate problem: Three loss terms (lemma 7)

In this section, I derive an approximation to the objective function with three loss terms: deviations from complete markets, price dispersion, and output gaps. I also show that the only difference between the global and the home planner is how much they care about insurance and the transfers that they desire. The proof uses lemma 6, which is presented in section B.2.4.

Lemma 7. *Around the deterministic steady state, the planner's objective function is approximately given by*

$$\mathcal{W} = -\frac{1}{2}\sum_s \pi_s \left\{ \left(A_{\mathcal{T}\mathcal{T}} + \gamma^* \frac{1}{m}\left(\frac{\partial U}{\partial C_T} - \bar{\lambda}\frac{dU^*}{dC^*} \right) \right) (\mathcal{T}_s - \mathcal{T}_{cm,s})^2 + \frac{1}{2}\lambda_\pi p_{Ns}^2 + \lambda_x x_s^2 \right\} + t.i.p. + \mathcal{O}(\epsilon^3) \tag{126}$$

where $\mathcal{T}_s = \sum_j \bar{\Theta}_j rr_{js}$ *is the total transfer received by the home country from abroad in state* s *(*rr_{js} *is the realized excess return of asset* j *relative to asset* 0 *in state* s*),* $\mathcal{T}_{cm,s}$ *are the transfers the planner would choose under complete markets and flexible prices given by:*

$$\mathcal{T}_{cm,s} = \left(A_{\mathcal{T}\mathcal{T}} + \gamma^* \frac{1}{m}\left(\frac{\partial U}{\partial C_T} - \bar{\lambda}\frac{dU^*}{dC^*} \right) \right)^{-1} A_{\mathcal{T}\xi}\xi_s \tag{127}$$

x_s *is the output gap, and* p_{Ns} *is the non-tradable price index. The constants are given by:*

$$A_{\mathcal{T}\mathcal{T}} = \frac{\partial U}{\partial C}\left(\gamma_T + \frac{1}{m}\gamma^* \right) > 0 \tag{128}$$

$$A_{\mathcal{T}\xi} = A_{c\xi} + A_{ll}^{-1} A_{cl} A_{l\xi} + \frac{\partial U}{\partial C_T}\left(-\gamma_T D_\xi Y_T + \gamma^* D_\xi Y^* \right) \tag{129}$$

$$\lambda_x = \bar{C}_N^2 \left(\frac{\partial F}{\partial L}\right)^{-2} A_{ll} > 0 \tag{130}$$

$$\lambda_\pi = \left(\frac{\phi}{1-\phi} \right) A_{pp} > 0 \tag{131}$$

$$\gamma_T = -\left(\frac{\partial U}{\partial C} \right)^{-1} \left(\frac{\partial^2 U}{\partial C_T^2} + A_{ll}^{-1} A_{cl}^2 \right) > 0 \tag{132}$$

$$\gamma^* = -\left(\frac{dU^*}{dC^*} \right)^{-1} \frac{d^2 U}{dC_T^{*2}} > 0. \tag{133}$$

Note that a global planner ($\frac{\frac{\partial U}{\partial C_T}}{\frac{dU^*}{dC^*}} = \bar{\lambda}$*) would put a smaller weight on the insurance term and choose larger transfers under complete markets than a home planner (*$\bar{\lambda} = 0$*).*

If prices were flexible, then $\int_0^1 (p_{Ns}(i) - p_{Ns})^2 di = 0$ and maximizing (117) with respect to l_s would yield the flexible-price allocation given some c_{Ts},

$$\bar{L} l_s^{\text{flex}} = A_{ll}^{-1} A_{cl} \bar{C}_T c_{Ts} + A_{ll}^{-1} A_{l\xi} \cdot \xi_s + \mathcal{O}(\epsilon^2). \tag{134}$$

Using this, I can rewrite (117) in terms of the output gap, $x_s = \bar{C}_N^{-1}(C_{Ns}^{\text{flex}} - C_{Ns})$ (given c_{Ts}), which is to first order given by

$$x_s = \bar{C}_N^{-1} \frac{\partial F}{\partial L} \bar{L}(l_s - l_s^{\text{flex}}) + \mathcal{O}(\epsilon^2). \tag{135}$$

This yields

$$U(C_{Ts}, C_{Ns}, L_s; \xi_s) = \frac{\partial U}{\partial C_T}\bar{C}_T(c_{Ts} + \frac{1}{2}c_{Ts}^2) + \frac{1}{2}\left(\frac{\partial^2 U}{\partial C_T^2} + \frac{A_{cl}^2}{A_{ll}}\right)\bar{C}_T^2 c_{Ts}^2 + \bar{C}_T c_{Ts}\left(A_{c\xi} + \frac{A_{cl}}{A_{ll}}A_{l\xi}\right)\cdot \xi_s \tag{136}$$
$$- \frac{1}{2}A_{pp}\left(\int_0^1 (p_{Ns}(i) - p_{Ns})^2 di\right) - \frac{1}{2}\lambda_x x_s^2 + \text{t.i.p.} + \mathcal{O}(\epsilon^3),$$

where λ_x is given by (130).

A first-order expansion of the price index (108) yields

$$p_{Ns}(i) = \frac{1}{1-\phi}p_{Ns} + \mathcal{O}(\epsilon^2) \text{ for } i \in \{\text{flex}\}. \tag{137}$$

Using this, I can rewrite (136) as

$$U(C_{Ts}, C_{Ns}, L_s; \xi_s) = \frac{\partial U}{\partial C_T}\bar{C}_T(c_{Ts} + \frac{1}{2}c_{Ts}^2) + \frac{1}{2}\left(\frac{\partial^2 U}{\partial C_T^2} + \frac{A_{cl}^2}{A_{ll}}\right)\bar{C}_T^2 c_{Ts}^2 \tag{138}$$
$$- \frac{1}{2}\lambda_\pi p_{Ns}^2 - \frac{1}{2}\lambda_x x_s^2 + \bar{C}_T c_{Ts}\left(A_{c\xi} + \frac{A_{cl}}{A_{ll}}A_{l\xi}\right)\cdot \xi_s + \text{t.i.p.} + \mathcal{O}(\epsilon^3),$$

where λ_π is given by (131).

A second-order approximation of the country's budget constraint (114) yields

$$\bar{C}_T c_{Ts} + \frac{1}{2}\bar{C}_T c_{Ts}^2 = D_\xi Y_T \cdot \xi_s + \frac{1}{2}\xi_s' D_{\xi\xi}^2 Y_T \cdot \xi_s + \mathcal{T}_s + \mathcal{O}(\epsilon^3) \tag{139}$$

Adding (138) over states, using (139), and discarding higher-order terms yields

$$\sum_s \pi_s U(C_{Ts}, C_{Ns}, L_s; \xi_s) = \sum_s \pi_s \Bigg\{ \frac{\partial U}{\partial C_T}\mathcal{T}_s + \frac{1}{2}\left(\frac{\partial^2 U}{\partial C_T^2} + \frac{A_{cl}^2}{A_{ll}}\right)\mathcal{T}_s^2 - \frac{1}{2}\lambda_\pi p_{Ns}^2 - \frac{1}{2}\lambda_x x_s^2$$
$$+ \mathcal{T}_s\left(A_{c\xi} + \frac{A_{cl}}{A_{ll}}A_{l\xi} + \left(\frac{\partial^2 U}{\partial C_T^2} + \frac{A_{cl}^2}{A_{ll}}\right)D_\xi Y_T\right)\cdot \xi_s \Bigg\} + \text{t.i.p.} + \mathcal{O}(\epsilon^3)$$

Next, add a second-order approximation of foreigners' flow utility with weight $m\bar{\lambda}$:

$$\mathcal{W} = \sum_s \pi_s \Bigg\{ \frac{\partial U}{\partial C_T}\mathcal{T}_s + \frac{1}{2}\left(\frac{\partial^2 U}{\partial C_T^2} + \frac{A_{cl}^2}{A_{ll}}\right)\mathcal{T}_s^2 + \mathcal{T}_s\left(A_{c\xi} + \frac{A_{cl}}{A_{ll}}A_{l\xi} + \left(\frac{\partial^2 U}{\partial C_T^2} + \frac{A_{cl}^2}{A_{ll}}\right)D_\xi Y_T\right)\cdot \xi_s$$
$$- \frac{1}{2}\lambda_\pi p_{Ns}^2 - \frac{1}{2}\lambda_x x_s^2 + m\bar{\lambda}\frac{dU^*}{dC^*}\bar{C}^*\left(c_s^* + \frac{1}{2}c_s^{*2}\right) + \frac{1}{2}m\bar{\lambda}\frac{d^2 U}{dC_T^{*2}}\bar{C}^{*2}c_s^{*2} \Bigg\} + \text{t.i.p.} + \mathcal{O}(\epsilon^3)$$

A second-order approximation to foreigners' budget constraint yields

$$\bar{C}^* c_s^* + \frac{1}{2}\bar{C}^* c_s^{*2} = D_\xi Y^* \cdot \xi_s + \xi_s' D_{\xi\xi}^2 Y^* \cdot \xi_s - \frac{1}{m}\mathcal{T}_s + \mathcal{O}(\epsilon^3).$$

Replacing back and discarding higher-order terms,

$$\mathcal{W} = \sum_s \pi_s \Bigg\{ \left(\frac{\frac{\partial U}{\partial C_T}}{\frac{dU^*}{dC^*}} - \bar{\lambda} \right) \frac{dU^*}{dC^*} \mathcal{T}_s - \frac{1}{2} A_{\mathcal{T}\mathcal{T}} \mathcal{T}_s^2 + \mathcal{T}_s A_{\mathcal{T}\xi} \cdot \xi_s - \frac{1}{2} \lambda_\pi p_{Ns}^2 - \frac{1}{2} \lambda_x x_s^2$$
$$+ \left(\bar{\lambda} - \frac{\frac{\partial U}{\partial C_T}}{\frac{dU^*}{dC^*}} \right) \frac{d^2 U}{dC^{*2}} \left(\frac{1}{2} \frac{1}{m} \mathcal{T}_s^2 - \mathcal{T}_s D_\xi Y^* \cdot \xi_s \right) \Bigg\} + \text{t.i.p.} + \mathcal{O}(\epsilon^3)$$

where $A_{\mathcal{T}\mathcal{T}}$ and $A_{\mathcal{T}\xi}$ are given by (128) and (129), respectively. A second-order approximation to the foreign no-arbitrage condition yields,

$$\sum_s \pi_s \mathcal{T}_s \frac{dU^*}{dC^*} + \sum_s \pi_s \mathcal{T}_s \frac{d^2 U^*}{dC^{*2}} \left(D_\xi Y^* \cdot \xi_s - \frac{1}{m} \mathcal{T}_s \right) = \mathcal{O}(\epsilon^3).$$

Thus,

$$\mathcal{W} = \sum_s \pi_s \Bigg\{ - \frac{1}{2} \left(A_{\mathcal{T}\mathcal{T}} + \frac{1}{m} \gamma^* \left(\frac{\partial U}{\partial C_T} - \frac{dU^*}{dC^*} \bar{\lambda} \right) \right) \mathcal{T}_s^2 + \mathcal{T}_s A_{\mathcal{T}\xi} \cdot \xi_s \tag{140}$$
$$- \frac{1}{2} \lambda_\pi p_{Ns}^2 - \frac{1}{2} \lambda_x x_s^2 \Bigg\} + \text{t.i.p.} + \mathcal{O}(\epsilon^3)$$

When prices are flexible and markets are complete, the optimal transfer maximizes (140). Solving this problem yields (127). Using the expression for $\mathcal{T}_{cm,s}$, one can rewrite (140) as desired.

B.2.6 Approximate problem: Two exchange rate targets (lemma 8)

Next, I prove that the objective function can be written as in the baseline model with two loss terms and two targets: a demand-management target, and an insurance target. The proof uses lemma 7, which is presented in section B.2.5.

Lemma 8. *Around the deterministic steady state, the planner's objective function is approximately given by*

$$\mathcal{W} = -\frac{1}{2} k_0 \left(\frac{1 - \sum_j \mu_j \bar{\Theta}_j}{1 - \sum_j \bar{R}_j \bar{\Theta}_j \frac{\partial X_j}{\partial C_T}} \right)^2 \sum_s \pi_s \left\{ \left(e_s - e_{dm,s}(\bar{\Theta}) \right)^2 + \chi f(\bar{\Theta})^2 \left(e_s - e_{in,s}(\bar{\Theta}) \right)^2 \right\} + t.i.p. + \mathcal{O}(\epsilon^3) \tag{141}$$

where

$$e_{dm,s}(\bar{\Theta}) = \frac{1 - \sum_j \bar{R}_j \bar{\Theta}_j \frac{\partial X_j}{\partial C_T}}{1 - \sum_j \mu_j \bar{R}_j \bar{\Theta}_j} \left(k_{ek} + k_{ec} D_\xi Y_T + k_{ec} \frac{\sum_j \bar{R}_j \bar{\Theta}_j \left(D_\xi X_j + \frac{\partial X_j}{\partial C_T} D_\xi Y_T \right)}{1 - \sum_j \bar{R}_j \bar{\Theta}_j \frac{\partial X_j}{\partial C_T}} \right) \cdot \xi_s \tag{142}$$

is the demand-management target, which attains a zero output gap and zero price dispersion when $\Theta = \bar{\Theta}$*,*

$$e_{in,s}(\bar{\Theta}) = \left(\frac{1 - \sum_j \bar{R}_j \bar{\Theta}_j \frac{\partial X_j}{\partial C_T}}{\sum_j \bar{R}_j \bar{\Theta}_j \frac{\partial X_j}{\partial E} \bar{E}} \right) \mathcal{T}_{cm,s} - \frac{\sum_j \bar{R}_j \bar{\Theta}_j \left(D_\xi X_j + \frac{\partial X_j}{\partial C_T} D_\xi Y_T \right)}{\sum_j \bar{R}_j \bar{\Theta}_j \frac{\partial X_j}{\partial E} \bar{E}} \cdot \xi_s \tag{143}$$

is the insurance target, which replicates the complete-market transfers when $\Theta = \bar{\Theta}$*, the balance-sheet* *exposure to monetary policy $f(\bar{\Theta})$ is given by*

$$f(\bar{\Theta}) = \frac{\sum_j k_{rr_je}\bar{\Theta}_j}{1-\sum_j \mu_j\bar{\Theta}_j},$$

where $k_{rr_je} = \bar{R}_j\frac{\partial X_j}{\partial E_j}\bar{E}$ is the response of the return of asset j when the exchange rate changes, and

$$k_0 = (\kappa^2\lambda_\pi + \lambda_x)(\kappa + k_{ex})^{-2} > 0$$

$$\chi = \left(A_{\mathcal{T}\mathcal{T}} + \frac{1}{m}\gamma^*\left(\frac{\partial U}{\partial C_T} - \frac{dU^*}{dC^*}\bar{\lambda}\right)\right)(\kappa^2\lambda_\pi + \lambda_x)^{-1}(\kappa + k_{ex})^2 > 0$$

$$\kappa = \left(1 - \eta\bar{C}_N(\frac{\partial F}{\partial L})^{-2}\frac{\partial^2 F}{\partial L^2}\right)^{-1}\frac{1-\phi}{\phi}(\frac{\partial F}{\partial L})^{-2}(\frac{\partial U}{\partial C_N})^{-1}\bar{C}_N A_{ll} > 0 \tag{144}$$

$$\mu_j = \bar{R}_j\left(\frac{\partial X_j}{\partial C_T} + k_{ec}\frac{\partial X_j}{\partial E}\bar{E}\right), \tag{145}$$

where k_{ec} and k_{ex} are given by (147) and (148), respectively.

A first-order approximation of (105) yields

$$e_s = p_{Ns} + k_{1c}\bar{C}_T c_{Ts} + k_{1l}\bar{L}l_s + k_{1\xi}\cdot\xi_s + \mathcal{O}(\epsilon^2)$$

where

$$k_{1c} \equiv -(\frac{\partial U}{\partial C_N})^{-1}\frac{\partial^2 U}{\partial C_N \partial C_T} + (\frac{\partial U}{\partial C_T})^{-1}\frac{\partial^2 U}{\partial C_T^2}$$

$$k_{1l} \equiv -(\frac{\partial U}{\partial C_N})^{-1}\frac{\partial^2 U}{\partial C_N^2}\frac{\partial F}{\partial L} - (\frac{\partial U}{\partial C_N})^{-1}\frac{\partial^2 U}{\partial C_N \partial L} + (\frac{\partial U}{\partial C_T})^{-1}\frac{\partial^2 U}{\partial C_T \partial C_N}\frac{\partial F}{\partial L} + (\frac{\partial U}{\partial C_T})^{-1}\frac{\partial^2 U}{\partial C_T \partial L}$$

$$k_{1\xi} \equiv -(\frac{\partial U}{\partial C_N})^{-1}\frac{\partial^2 U}{\partial C_N^2}D_\xi F - (\frac{\partial U}{\partial C_N})^{-1}D^2_{C_N\xi}U + (\frac{\partial U}{\partial C_T})^{-1}\frac{\partial^2 U}{\partial C_T \partial C_N}D_\xi F + (\frac{\partial U}{\partial C_T})^{-1}D^2_{C_T\xi}U.$$

Using (134), I can rewrite this in terms of output gap deviations (given c_{Ts}):

$$e_s = p_{Ns} + k_{ec}\bar{C}_T c_{Ts} + k_{e\xi}\cdot\xi_s + k_{ex}x_s + \mathcal{O}(\epsilon^2) \tag{146}$$

where

$$k_{ec} \equiv k_{1c} + k_{1l}A_{ll}^{-1}A_{cl} \tag{147}$$

$$k_{e\xi} \equiv k_{1\xi} + k_{1l}A_{ll}^{-1}A_{l\xi}$$

$$k_{ex} \equiv \bar{C}_N(\frac{\partial F}{\partial L})^{-1}k_{1l}. \tag{148}$$

A first-order expansion of the definition of realized excess returns of asset j relative to asset 0 in state s yields:

$$rr_{js} = \bar{R}_j\bar{X}_j r_j + \bar{R}_j\frac{\partial X_j}{\partial C_{Ts}}\bar{C}_T c_{Ts} + \bar{R}_j\frac{\partial X_j}{\partial E_s}\bar{E}e_s + \bar{R}_j D_\xi X_j\cdot\xi_s + \mathcal{O}(\epsilon^2). \tag{149}$$

Assuming all shocks are mean zero and noting that the planner would never deviate in expectation from demand-management,[51]

$$rr_{js} = \bar{R}_j \frac{\partial X_j}{\partial C_{Ts}} \bar{C}_T c_{Ts} + \bar{R}_j \frac{\partial X_j}{\partial E_s} \bar{E} e_s + \bar{R}_j D_\xi X_j \cdot \xi_s + \mathcal{O}(\epsilon^2).$$

Next, multiply each equation j by the steady-state position $\bar{\Theta}_j$ and add over j to obtain

$$\underbrace{\sum_j rr_{js} \bar{\Theta}_j}_{=\mathcal{T}_s} = \sum_j \bar{R}_j \bar{\Theta}_j \frac{\partial X_j}{\partial C_{Ts}} \bar{C}_T c_{Ts} + \bar{R}_j \bar{\Theta}_j \frac{\partial X_j}{\partial E_s} \bar{E} e_s + \bar{R}_j \bar{\Theta}_j D_\xi X_j \cdot \xi_{ks} + \mathcal{O}(\epsilon^2).$$

Using a first-order approximation of the country's budget constraint (114) and solving for $\mathcal{T}_s$ yields

$$\mathcal{T}_s = \frac{1}{1 - \sum_j \bar{R}_j \bar{\Theta}_j \frac{\partial X_j}{\partial C_T}} \sum_j \left(\bar{R}_j \bar{\Theta}_j \frac{\partial X_j}{\partial E} \bar{E} e_s + \bar{R}_j \bar{\Theta}_j \left(D_\xi X_j + \frac{\partial X_j}{\partial C_T} D_\xi Y_T \right) \cdot \xi_s \right) + \mathcal{O}(\epsilon^2) \qquad (150)$$

Replacing this expression into (146) and solving for e_s when $p_{Ns} = 0$ and $x_s = 0$ yields the demand-management target (142).

Using this, (146) can be rewritten as

$$e_s = \frac{(1 - \sum_j \bar{R}_j \bar{\Theta}_j \frac{\partial X_j}{\partial C_T})}{1 - \sum_j \mu_j \bar{\Theta}_j} (p_{Ns} + k_{ex} x_s) + e_{dm,s}(\bar{\Theta}) + \mathcal{O}(\epsilon^2). \qquad (151)$$

Combining (105) and (106),

$$(-\frac{\partial U}{\partial L_s}(s)) / \frac{\partial U}{\partial C_{Ns}}(s) = \frac{W_s}{P_{Ns}}.$$

Approximate this to first-order to obtain:

$$p_{Ns} - w_s = k_{2c} \bar{C}_T c_{Ts} + k_{2l} \bar{L} l_s + k_{2\xi} \cdot \xi_s + \mathcal{O}(\epsilon^2) \qquad (152)$$

where

$$k_{2c} \equiv (\frac{\partial U}{\partial C_N})^{-1} \frac{\partial^2 U}{\partial C_N \partial C_T} + (\frac{\partial U}{\partial C_N})^{-1} (\frac{\partial F}{\partial L})^{-1} \frac{\partial^2 U}{\partial L \partial C_T}$$

$$k_{2l} \equiv (\frac{\partial U}{\partial C_N})^{-1} \frac{\partial^2 U}{\partial C_N^2} \frac{\partial F}{\partial L} + 2(\frac{\partial U}{\partial C_N})^{-1} \frac{\partial^2 U}{\partial C_N \partial L} + (\frac{\partial U}{\partial C_N})^{-1} (\frac{\partial F}{\partial L})^{-1} \frac{\partial^2 U}{\partial L^2}$$

$$k_{2\xi} \equiv (\frac{\partial U}{\partial C_N})^{-1} \frac{\partial^2 U}{\partial C_N^2} D_\xi F + (\frac{\partial U}{\partial C_N})^{-1} D^2_{C_N \xi} U + (\frac{\partial U}{\partial C_N})^{-1} (\frac{\partial F}{\partial L})^{-1} \frac{\partial^2 U}{\partial C_N \partial L} D_\xi F + (\frac{\partial U}{\partial C_N})^{-1} (\frac{\partial F}{\partial L})^{-1} D^2_{L\xi} U.$$

A first-order approximation of (113) yields for $i \in$ flex,

$$p_{Ns}(i) + (\frac{\partial F}{\partial L})^{-1} \frac{\partial^2 F}{\partial L^2} \bar{L} l_s(i) + (\frac{\partial F}{\partial L})^{-1} D^2_{L\xi} F \cdot \xi_s = w_s + \mathcal{O}(\epsilon^2).$$

[51] If shocks were not mean zero, then the following expressions would hold in terms of the innovations, i.e. $\tilde{y}_s = y_s - \mathbb{E} y_s$ for any variable y. The remainder would be unchanged. See appendix B.4 for an example with infinite horizon where shocks at $t+1$ conditional on the time-t information set are not mean zero and, hence, innovations are considered explicitly.

Next, use (152) to get rid of w_s,

$$p_{Ns}(i) - p_{Ns} + (\frac{\partial F}{\partial L})^{-1} \frac{\partial^2 F}{\partial L^2} \bar{L} \left(l_s(i) - l_s\right) + (\frac{\partial U}{\partial C_N})^{-1} (\frac{\partial F}{\partial L})^{-1} \left(A_{cl} \bar{C}_T c_{Ts} - A_{ll} \bar{L} l_s + A_{l\xi} \cdot \xi_s\right) = \mathcal{O}(\epsilon^2)$$

In terms of the output gap x_s,

$$p_{Ns}(i) - p_{Ns} + (\frac{\partial F}{\partial L})^{-1} \frac{\partial^2 F}{\partial L^2} \bar{L} \left(l_s(i) - l_s\right) - (\frac{\partial F}{\partial L})^{-2} (\frac{\partial U}{\partial C_N})^{-1} \bar{C}_N A_{ll} x_s = \mathcal{O}(\epsilon^2).$$

A first-order approximation of demand (107), and production (111),

$$\bar{C}_N^{-1} \bar{L} \frac{\partial F}{\partial L} l_s(i) = -\eta(p_{Ns}(i) - p_{Ns}) + \bar{C}_N^{-1} \bar{L} \frac{\partial F}{\partial L} l_s$$

Using the first-order expansion of the price index (137) and replacing,

$$p_{Ns} = \kappa x_s \tag{153}$$

where κ is given by (144). Using (153), I can solve for x_s from (151):

$$x_s = \left(\frac{1}{\kappa + k_{ex}}\right) \left(\frac{1 - \sum_j \mu_j \bar{\Theta}_j}{1 - \sum_j \bar{R}_j \frac{\partial X_j}{\partial C_T} \bar{\Theta}_j}\right) \left(e_s - e_{dm,s}(\bar{\Theta})\right) + \mathcal{O}(\epsilon^2). \tag{154}$$

When the planner cares only about insurance, it chooses $e_{in,s}$ such that (150) is equal to $\mathcal{T}_{cm,s}$:

$$e_{in,s}(\bar{\Theta}) = \frac{\left(1 - \sum_j \bar{R}_j \frac{\partial X_j}{\partial C_T} \bar{\Theta}_j\right) \mathcal{T}_{cm,s} - \sum_j \bar{R}_j \left(D_\xi X_j + \frac{\partial X_j}{\partial C_T} D_\xi Y_T\right) \bar{\Theta}_j \cdot \xi_s}{\sum_j \bar{R}_j \frac{\partial X_j}{\partial E} \bar{\Theta}_j \bar{E}} + \mathcal{O}(\epsilon^2).$$

Using this, the insurance loss term can be written as

$$\mathcal{T}_s - \mathcal{T}_{cm,s} = \frac{1}{1 - \sum_j \bar{R}_j \frac{\partial X_j}{\partial C_T} \bar{\Theta}_j} \sum_j \bar{R}_j \frac{\partial X_j}{\partial E} \bar{\Theta}_j \bar{E} \left(e_s - e_{in,s}(\bar{\Theta})\right) + \mathcal{O}(\epsilon^2). \tag{155}$$

Putting (154) and (155) together and rearranging yields (141).

B.2.7 Proof of proposition 10

The optimal exchange rate is still given by

$$e_s = \frac{\chi f(\bar{\Theta})^2}{1 + \chi f(\bar{\Theta})^2} e_{in,s}(\bar{\Theta}) + \frac{1}{1 + \chi f(\bar{\Theta})^2} e_{dm,s}(\bar{\Theta}). \tag{156}$$

Replacing this expression into (141) yields after some algebra,

$$\mathcal{W} = -\frac{1}{2} k_0 \left(\frac{\chi}{1 + \chi f(\bar{\Theta})^2}\right) \left(\frac{\sum_j \bar{R}_j \frac{\partial X_j}{\partial E} \bar{\Theta}_j \bar{E}}{1 - \sum_j \bar{R}_j \frac{\partial X_j}{\partial C_T} \bar{\Theta}_j}\right)^2 \sum_s \pi_s \left\{\left(e_{in,s}(\bar{\Theta}) - e_{dm,s}(\bar{\Theta})\right)^2\right\} + \text{t.i.p.} + \mathcal{O}(\epsilon^3)$$

Using (150) and (146) evaluated at $p_{Ns} = x_s = 0$, one can show that

$$\mathcal{T}_{dm,s}(\bar{\Theta}) = \frac{1}{1 - \sum_j \mu_j \bar{\Theta}_j} \sum_j \bar{\Theta}_j rr_{dm,js}(0)$$

where $\mathcal{T}_{dm,s}(\bar{\Theta})$ is the equilibrium transfer when $e = e_{dm,s}(\bar{\Theta})$ and $\Theta = \bar{\Theta}$ and $rr_{dm,js}$ are the realized excess returns under demand management when $\bar{\Theta} = 0$,

$$rr_{dm,js}(0) = \bar{R}_j \frac{\partial X_j}{\partial E} \bar{E} e_{dm,s}(0) + \bar{R}_j \left(D_\xi X_j + \frac{\partial X_j}{\partial C_T} D_\xi Y_T \right) \cdot \xi_s.$$

Using (150) evaluated at $e_s = e_{in,s}(\bar{\Theta})$ and subtracting,

$$e_{in,s}(\bar{\Theta}) - e_{dm,s}(\bar{\Theta}) = \left(\frac{1 - \sum_j \bar{R}_j \frac{\partial X_j}{\partial C_T} \bar{\Theta}_j}{\sum_j \bar{R}_j \frac{\partial X_j}{\partial E} \bar{\Theta}_j \bar{E}} \right) \left(\mathcal{T}_{cm,s} - \frac{1}{1 - \sum_j \mu_j \bar{\Theta}_j} \sum_j \bar{\Theta}_j rr_{dm,js}(0) \right).$$

Define $\tilde{\Theta}_j = \left(\frac{1}{1 - \sum_j \mu_j \bar{\Theta}_j} \right) \bar{\Theta}_j$. The objective becomes

$$-\frac{1}{2} k_0 \left(\frac{\chi}{1 + \chi f(\bar{\Theta})^2} \right) \left(\sigma^2_{\mathcal{T}_{cm}} + \tilde{\Theta}' \text{Var}(rr_{dm}(0)) \tilde{\Theta} - 2\tilde{\Theta}' \text{Cov}(\mathcal{T}_{cm}, rr_{dm}(0)) \right). \tag{157}$$

First, I solve for the optimal portfolio $\{\tilde{\Theta}_j\}_j$ given some balance-sheet exposure to monetary policy $f(\bar{\Theta})$. Let $k_{rre} = \{\bar{R}_j \frac{\partial X_j}{\partial E} \bar{E}\}_j \in \mathbb{R}^J$ and η denote the multiplier of the constraint $\tilde{\Theta}' k_{rre} = f(\bar{\Theta})$. The FOC with respect to $\tilde{\Theta}_j$ yields

$$\tilde{\Theta} = \text{Var}(rr_{dm}(0))^{-1} \left\{ \eta k_{rre} + \text{Cov}(\mathcal{T}_{cm}, rr_{dm}(0)) \right\}.$$

Replacing in the constraint and solving,

$$\eta = \frac{1}{k'_{rre} \text{Var}(rr_{dm}(0))^{-1} k_{rre}} \left\{ f(\bar{\Theta}) - k'_{rre} \text{Var}(rr_{dm}(0))^{-1} \text{Cov}(\mathcal{T}_{cm}, rr_{dm}(0)) \right\}.$$

Replacing back,

$$\tilde{\Theta} = k_{\Theta 0} + k_{\Theta f} f(\bar{\Theta}) \tag{158}$$

where

$$k_{\Theta 0} = \left(I - \frac{\text{Var}(rr_{dm}(0))^{-1} k_{rre} k'_{rre}}{k'_{rre} \text{Var}(rr_{dm}(0))^{-1} k_{rre}} \right) \text{Var}(rr_{dm}(0))^{-1} \text{Cov}(\mathcal{T}_{cm}, rr_{dm}(0))$$

$$k_{\Theta f} = \frac{\text{Var}(rr_{dm}(0))^{-1} k_{rre}}{k'_{rre} \text{Var}(rr_{dm}(0))^{-1} k_{rre}}.$$

Next, suppose that $k_{rr_j e} = 0 \; \forall j$. Then, only $f(\bar{\Theta}) = 0$ is feasible. The FOC without the $f(\bar{\Theta})$ constraint yields the desired result.

Note that one can recover the original portfolios $\bar{\Theta}_j$ by solving the fixed point in the definition of $\tilde{\Theta}_j$,

$$\bar{\Theta}_j = \frac{1}{1 + \sum_j \mu_j \left(k_{\Theta_j 0} + k_{\Theta_j f} f(\bar{\Theta}) \right)} \left(k_{\Theta_j 0} + k_{\Theta_j f} f(\bar{\Theta}) \right).$$

B.2.8 Proof of lemma 5

Replacing (158) into (157) yields this result.

B.2.9 Proof of proposition 11

The robustness of propositions 2, 3, and 4 in terms of $f(\bar{\Theta})$ is immediate from lemma 5. Furthermore, note that using the definitions of $\mathcal{T}_{dm,s}$ and $\mathcal{T}_{cm,s}$, the optimal exchange rate rule (156) can be written as

$$\mathcal{T}_s = \frac{\chi f(\bar{\Theta})^2}{1+\chi f(\bar{\Theta})^2}\mathcal{T}_{cm,s} + \frac{1}{1+\chi f(\bar{\Theta})^2}\sum_j \tilde{\Theta}_j rr_{dm,js}(0).$$

Replacing the optimal portfolios $\tilde{\Theta}_j$, given by (158), and rr_{fs}, given by (35),

$$rr_{fs} = \frac{\chi f(\bar{\Theta})^2}{1+\chi f(\bar{\Theta})^2} f(\bar{\Theta})^{-1}\tilde{\mathcal{T}}_{cm,s} + \frac{1}{1+\chi f(\bar{\Theta})^2}\tilde{rr}_{dm,s}(0).$$

The proof of the result for the volatility of rr_{fs} is identical to that in proposition 7.

B.2.10 Proof of proposition 12

Using equations (146) and (153) to replace e_s and p_{Ns} in the expansion of realized returns (149),

$$rr_{js} = k_{rr_j\mathcal{T}}\mathcal{T}_s + k_{rr_jx}x_s + k_{rr_j\xi}\cdot\xi_s + \mathcal{O}(\epsilon^2)$$

where

$$\begin{aligned}
k_{rr_j\mathcal{T}} &= \bar{R}_j\frac{\partial X_j}{\partial C_{Ts}} + \bar{R}_j k_{ec}\frac{\partial X_j}{\partial E_s}\bar{E} \\
k_{rr_jx} &= \bar{R}_j\left(\kappa + k_{ex}\right)\frac{\partial X_j}{\partial E_s}\bar{E} \\
k_{rr_j\xi} &= \bar{R}_j\left(D_\xi X_j + \frac{\partial X_j}{\partial C_{Ts}}D_\xi Y_T + \left(k_{ek} + k_{ec}D_\xi Y_T\right)\frac{\partial X_j}{\partial E_s}\bar{E}\right).
\end{aligned}$$

Using lemma 7, the planner problem can be written as

$$\max_{\{\mathcal{T}_s,\{rr_{js}\}_j,x_s\}_s} -\sum_s \pi_s\left\{\frac{1}{2}\left(A_{\mathcal{T}\mathcal{T}} + \frac{1}{m}\gamma^*\left(\frac{\partial U}{\partial C_T} - \bar{\lambda}\frac{dU^*}{dC^*}\right)\right)\left(\mathcal{T}_s - \mathcal{T}_{cm,s}\right)^2 + \frac{1}{2}(\kappa\lambda_\pi + \lambda_x)x_s^2\right\} + \text{t.i.p.} + \mathcal{O}(\epsilon^3) \tag{159}$$

subject to

$$\begin{aligned}
\mathcal{T}_s &= \sum_j \bar{\Theta}_j rr_{js} \\
rr_{js} &= k_{rr_j\mathcal{T}}\mathcal{T}_s + k_{rr_jx}x_s + k_{rr_j\xi}\cdot\xi_s.
\end{aligned}$$

The first-order conditions yield

$$-\left(A_{\mathcal{T}\mathcal{T}}+\frac{1}{m}\gamma^*\left(\frac{\partial U}{\partial C_T}-\bar{\lambda}\frac{dU^*}{dC^*}\right)\right)(\mathcal{T}_s-\mathcal{T}_{cm,s})+\lambda_s+\sum_j \zeta_{js}k_{rr_j\mathcal{T}}=0 \tag{160}$$

$$-(\kappa\lambda_\pi+\lambda_x)x_s+\sum_j \zeta_{js}k_{rr_jx}=0 \tag{161}$$

$$-\bar{\Theta}_j\lambda_s+\zeta_{js}=0 \tag{162}$$

$$\sum_s \pi_s\lambda_s rr_{js}=0 \tag{163}$$

Replacing (162) into (161),

$$(\kappa\lambda_\pi+\lambda_x)x_s=\left(\sum_j \bar{\Theta}_j k_{rr_jx}\right)\lambda_s.$$

Thus, using the portfolio optimality condition (163),

$$\sum_s \pi_s x_s rr_{js}=0. \tag{164}$$

This shows that the planner always chooses output gaps such that they are uncorrelated with the returns of the assets.

Replacing (162) into (160),

$$\left(A_{\mathcal{T}\mathcal{T}}+\frac{1}{m}\gamma^*\left(\frac{\partial U}{\partial C_T}-\bar{\lambda}\frac{dU^*}{dC^*}\right)\right)(\mathcal{T}_s-\mathcal{T}_{cm,s})=\left(1+\sum_j \bar{\Theta}_j k_{rr_j\mathcal{T}}\right)\lambda_s.$$

Then, using the portfolio optimality condition (163),

$$\sum_s \pi_s(\mathcal{T}_s-\mathcal{T}_{cm,s})rr_{js}=0.$$

This shows that the planner always chooses transfers such that they are uncorrelated with the returns of the assets. Using the definition of $\mathcal{T}_{cm,s}$ provided in lemma 7, one may rewrite this as

$$-\left(A_{\mathcal{T}\mathcal{T}}+\frac{1}{m}\gamma^*\left(\frac{\partial U}{\partial C_T}-\bar{\lambda}\frac{dU^*}{dC^*}\right)\right)\sum_s \pi_s\mathcal{T}_s rr_{js}+\sum_s \pi_s rr_{js}A_{\mathcal{T}\xi}\cdot\xi_s=0. \tag{165}$$

Next, consider the home and foreign no-arbitrage conditions,

$$\sum_s \pi_s\frac{\partial U}{\partial C_T}(s)(rr_{js}-\tau_j)=0$$

$$\sum_s \pi_s\frac{dU^*}{dC^*}(s)rr_{js}=0,$$

where $rr_{js}=R_jX_j-1$ is the excess realized return before taxes. Approximating both equations to second-order and combining them to get rid of the linear term in rr_{js} yields

$$-\sum_s \pi_s A_{\mathcal{T}\mathcal{T}} \mathcal{T}_s rr_{js} + \underbrace{\sum_s \pi_s \left(\frac{\partial F}{\partial L}\right)^{-1} \bar{C}_N A_{cl} rr_{js} x_s}_{=0} + \sum_s \pi_s rr_{js} A_{\mathcal{T}\xi} \cdot \xi_s = \frac{\partial U}{\partial C_T} \tau_j + \mathcal{O}(\epsilon^3).$$

Equation (164) implies the second term is zero. Using (165) and rearranging yields the desired result.

B.3 Dynamic model: Three period model

This appendix contains details of the three-period model of sections 5.1−5.2. Section B.3.1 formally defines the competitive equilibrium and the planning problem. Section B.3.2 derives a second-order approximation of the objective function (equation 42). Section B.3.2 derives a first-order approximation of the constraint (equation 44.). Sections B.3.4, B.3.5, B.3.6, and B.3.7 prove propositions 13, 14, 15, and 16 respectively.

B.3.1 Competitive equilibrium and planning problem

As argued in the main text, optimization yields conditions analogous to the ones in the static model. These hold for all s, t:

$$\frac{\alpha}{1-\alpha} \frac{C_{Nst}}{C_{Tst}} = E_{st}/P_{Nst} \tag{166}$$

$$\kappa^{-1} L_s^{\varphi} \frac{C_{Nst}^{\alpha}}{C_{Tst}^{\alpha}} = \frac{W_{st}}{P_{Nst}} \tag{167}$$

$$C_{Nst} = Z_{st} L_{st} \tag{168}$$

$$P_{Nst} = 1. \tag{169}$$

The home and foreign no-arbitrage equations, coming from asset optimization at $t = 0$, are

$$\sum_s \pi_s \left(rr_s - \tau_B\right) \frac{\partial U}{\partial C_T}(s, 1) = 0 \tag{170}$$

$$\sum_s \pi_s rr_s \frac{\partial U^*}{\partial C_T^*}(s, 1) = 0 \tag{171}$$

Next, I formally define the competitive equilibrium and the planner's problem.

Definition 4. Given a Central Bank policy ($\{E_{s1}, E_{s2}\}_s$,τ_B, $\{\tau_s^{sav}\}_s$), an allocation ($\{C_{Tst}\}_{s,t}$,$\{C_{Nst}\}_{s,t}$, $\{L_{s,t}\}_{s,t}$,$\{nfa_s\}$,B) together with prices ($\{P_{Nst}\}_s$,$\{W_{st}\}_s$,R_0, R_s) and realized excess returns $\{rr_s\}_s$, is a ***competitive equilibrium*** if and only if they solve (37) − (41) and (166)−(171).

Problem 4. The planner's problem is to choose ($\{E_{s1}, E_{s2}\}_s$,τ_B) to maximize (36) subject to (37) − (41) and (166)−(171).

B.3.2 Approximate welfare

A second-order approximation to flow utility yields

$$U(s,t) = \left(\frac{1+\varphi}{\alpha+\varphi}\right)\alpha c_{Tst} - \frac{1}{2}(1+\varphi)(1-\alpha)\left(e_{st} + \frac{\varphi}{\alpha+\varphi} c_{Tst} - \frac{1+\varphi}{\alpha+\varphi} z_{st}\right)^2 + \text{t.i.p.} + \mathcal{O}(\epsilon^3)$$

Using a first-order approximation of (11) and the definition of the output gap (135), this can be rewritten as

$$U(s,t) = (\frac{1+\varphi}{\alpha+\varphi})\alpha c_{Tst} - \frac{1}{2}(1+\varphi)(1-\alpha)^3 x_{st}^2 + \text{t.i.p.} + \mathcal{O}(\epsilon^3). \tag{172}$$

Approximating (40) and (41) to second order yields,[52]

$$\alpha c_{Ts1} + \frac{1}{2}\alpha c_{Ts1}^2 + nfa_s = \alpha y_{Ts1} + \frac{1}{2}\alpha y_{Ts1}^2 + \mathcal{T}_s + \mathcal{O}(\epsilon^3)$$
$$\alpha c_{Ts2} + \frac{1}{2}\alpha c_{Ts2}^2 = \alpha y_{Ts2} + \frac{1}{2}\alpha y_{Ts2}^2 + nfa_s + nfa_s r_s^* + \mathcal{O}(\epsilon^3)$$

Replacing in (172), adding both time periods and discarding higher-order terms yields

$$\mathcal{W} = \sum_s \pi_s \Bigg\{ \left(\frac{1+\varphi}{\alpha+\varphi}\right)\left(\mathcal{T}_s - \frac{1}{2}\alpha^{-1}\mathcal{T}_s^2 - \alpha^{-1} nfa_s^2 - y_{1s}\mathcal{T}_s + nfa_s\left(y_{T1s} - y_{T2s} + \alpha^{-1}\mathcal{T}_s + r_s^*\right)\right)$$
$$- \frac{1}{2}(1+\varphi)(1-\alpha)^3 \sum_{t=1,2} x_{st}^2 \Bigg\} + \text{t.i.p.} + \mathcal{O}(\epsilon^3)$$

Using a second-order approximation of the foreign no-arbitrage condition (171) and replacing,

$$\mathcal{W} = \sum_s \pi_s \Bigg\{ \left(\frac{1+\varphi}{\alpha+\varphi}\right)\left(\gamma^*\mathcal{T}_s c_s^* - \frac{1}{2}\alpha^{-1}\mathcal{T}_s^2 - \alpha^{-1} nfa_s^2 - y_{1s}\mathcal{T}_s + nfa_s\left(y_{T1s} - y_{T2s} + \alpha^{-1}\mathcal{T}_s + r_s^*\right)\right)$$
$$- \frac{1}{2}(1+\varphi)(1-\alpha)^3 \sum_{t=1,2} x_{st}^2 \Bigg\} + \text{t.i.p.} + \mathcal{O}(\epsilon^3)$$

If prices were flexible, maximizing over nfa_s would yield the optimal savings conditional on some transfer $\mathcal{T}_s$,

$$nfa_s^{\text{fb}}(\mathcal{T}_s) = \frac{1}{2}\left(\alpha(y_{1s} - y_{2s}) + \alpha r_s^* + \mathcal{T}_s\right).$$

Using this and defining $\tilde{nfa}_s = nfa_s - nfa_s^{\text{fb}}(\mathcal{T}_s)$, the objective function can be rewritten as

$$\mathcal{W} = \sum_s \pi_s \Bigg\{ \left(\frac{1+\varphi}{\alpha+\varphi}\right)\left(-\frac{1}{4}\alpha^{-1}\mathcal{T}_s^2 - \alpha^{-1}\tilde{nfa}_s^2 + \left(-\frac{1}{2}(y_{Ts1} + y_{Ts2}) + \frac{1}{2}r_s^* + \gamma^* c_s^*\right)\mathcal{T}_s\right)$$
$$- \frac{1}{2}(1+\varphi)(1-\alpha)^3 \sum_{t=1,2} x_{st}^2 \Bigg\} + \text{t.i.p.} + \mathcal{O}(\epsilon^3)$$

If prices were flexible, setting $\tilde{nfa}_s = x_{st} = 0\ \forall s,t$ and maximizing over $\mathcal{T}_s$ would yield the complete-markets transfers:

$$\mathcal{T}_{cm,s} = -\alpha(y_{s1} + y_{s2}) + \alpha r_s^* + 2\alpha\gamma^* c_s^*.$$

Armed with this, the objective function becomes

$$\mathcal{W} = -\sum_s \pi_s \left\{ \left(\frac{1+\varphi}{\alpha+\varphi}\right)\alpha^{-1}\left(\frac{1}{4}\left(\mathcal{T}_s - \mathcal{T}_{cm,s}\right)^2 + \tilde{nfa}_s^2\right) + \frac{1}{2}(1+\varphi)(1-\alpha)^3 \sum_{t=1,2} x_{st}^2 \right\} + \text{t.i.p.} + \mathcal{O}(\epsilon^3)$$

[52]Recall that, for ease of exposition, I assumed $\bar{nfa} = 0$.

Rearranging yields (42).

B.3.3 Approximate constraint

Using a first-order approximation of the budget constraints (40) and (41) to substitute out c_{Ts} in the definition of the output gap yields

$$(1-\alpha)x_{s1} = e_{s1} + \frac{\varphi}{\alpha+\varphi}\left(y_{Ts1} + \alpha^{-1}\bar{B}rr_s - \alpha^{-1}nfa_s\right) - \frac{1+\varphi}{\alpha+\varphi}z_{s1}$$
$$(1-\alpha)x_{s2} = e_{s2} + \frac{\varphi}{\alpha+\varphi}\left(y_{Ts2} + \alpha^{-1}nfa_s\right) - \frac{1+\varphi}{\alpha+\varphi}z_{s2}.$$

Assuming w.l.o.g. that shocks are mean zero and replacing into (43),

$$rr_s = \delta\left(\frac{\varphi}{\alpha+\varphi}\left(y_{Ts1} + \alpha^{-1}\bar{B}rr_s - \alpha^{-1}nfa_s\right) - \frac{1+\varphi}{\alpha+\varphi}z_{s1} - (1-\alpha)x_{s1}\right) \tag{173}$$
$$+ (1-\delta)\left(\frac{\varphi}{\alpha+\varphi}\left(y_{Ts2} + \alpha^{-1}nfa_s\right) - \frac{1+\varphi}{\alpha+\varphi}z_{s2} - (1-\alpha)x_{s2}\right) + (1-\delta)(\psi_s - r_s^*).$$

After some algebra, I can rewrite this in terms of $\tilde{nfa}_s = nfa_s - nfa_s^{\text{fb}}(0) - \frac{1}{2}\bar{B}rr_s$,

$$rr_s = rr_{dm,s}(0) - (1-\alpha)(\delta x_{s1} + (1-\delta)x_{s2}) + \frac{\alpha^{-1}\varphi}{\alpha+\varphi}\left(-\delta + (1-\delta)\right)\tilde{nfa}_s + \frac{1}{2}\frac{\alpha^{-1}\varphi}{\alpha+\varphi}\bar{B}rr_s,$$

where $rr_{dm,s}(0)$ is the solution to (173) when $\bar{B} = 0$, $nfa_s = nfa_s^{\text{fb}}(0)$, and $x_{s1} = x_{s2} = 0$ (equation 45). Rearranging yields (44).

B.3.4 Proof of proposition 13

The planner chooses $\{x_{s1}, x_{s2}, \tilde{nfa}_s\}$ to maximize (42) subject to (44). The solution is given by

$$\tilde{nfa}_s = -\left(\delta - \frac{1}{2}\right)\left(\frac{\tilde{\chi}^{-1}\mu}{\delta^2 + (1-\delta)^2 + \tilde{\chi}^{-1}\mu^2\left(\delta - (1-\delta)\right)^2}\right)\left(\left(1-\mu\bar{B}\right)rr_s - rr_{dm,s}(0)\right) \tag{174}$$
$$x_{s1} = -\frac{1}{1-\alpha}\left(\frac{\delta}{\delta^2 + (1-\delta)^2 + \tilde{\chi}^{-1}\mu^2\left(\delta - (1-\delta)\right)^2}\right)\left(\left(1-\mu\bar{B}\right)rr_s - rr_{dm,s}(0)\right) \tag{175}$$
$$x_{s2} = -\frac{1}{1-\alpha}\left(\frac{1-\delta}{\delta^2 + (1-\delta)^2 + \tilde{\chi}^{-1}\mu^2\left(\delta - (1-\delta)\right)^2}\right)\left(\left(1-\mu\bar{B}\right)rr_s - rr_{dm,s}(0)\right). \tag{176}$$

The results are immediate from these expressions.

B.3.5 Proof of proposition 14

Replacing (174) – (176) into (47) and solving for τ_s^{sav} yields

$$\tau_s^{\text{sav}} = 2\left(\delta - \frac{1}{2}\right)\left(\frac{1-\alpha+\alpha^{-1}\tilde{\chi}^{-1}\mu}{\delta^2 + (1-\delta)^2 + \tilde{\chi}^{-1}\mu^2(\delta - (1-\delta))^2}\right)\left(\left(1-\mu\bar{B}\right)rr_s - rr_{dm,s}(0)\right).$$

The results are immediate from this expression.

B.3.6 Proof of proposition 15

The fact that lemma 2 carries over in terms of $f(\bar{B})$ is immediate from the approximate objective (48). Replacing the optimal realized excess returns into (48) yields

$$\mathcal{W} = -\frac{1}{2}\frac{\chi k_0}{1+\chi f(\bar{B})^2}\left(\underbrace{f(\bar{B})^2\sigma^2_{rr_{dm}(0)}}_{\text{demand-management}} + \underbrace{\sigma^2_{\mathcal{T}_{cm}}}_{\text{insurance}} + \underbrace{2f(\bar{B})\sigma_{\mathcal{T}_{cm}rr_{dm}(0)}}_{\text{align targets}}\right) + \text{t.i.p.} + \mathcal{O}(\epsilon^3) \tag{177}$$

Since (177) is identical to (20) with $f(\bar{B})$, $\sigma^2_{e_{dm}(0)}$, and $\sigma_{\mathcal{T}_{cm}rr_{dm}(0)}$ instead of $\bar{B}$, $\sigma^2_{e_{dm}}$, and $\sigma_{\mathcal{T}_{cm}rr_{dm}(0)}$, respectively, the extension of propositions 2, 3 and 4 is immediate.

Rewriting the optimal returns (29) in terms of rr_{fs},

$$rr_{fs} = (1-\omega)rr_{dm,s}(0) + \frac{1}{f(\bar{B})}\omega\mathcal{T}_{cm,s} + \mathcal{O}(\epsilon^2). \tag{178}$$

Following steps analogous to the proof of lemma 3 (see appendix B.1.4),

$$\frac{\partial\sigma^2_{rr_f}}{\partial\omega} = \frac{2}{f(\bar{B})}\frac{1}{1+\chi f(\bar{B})^2}\left(\chi f(\bar{B})^2\sigma_{\mathcal{T}_{cm}rr_{dm}(0)} + \left(\chi\sigma^2_{\mathcal{T}_{cm}} - \sigma^2_{rr_{dm}(0)}\right)f(\bar{B}) - \sigma_{\mathcal{T}_{cm}rr_{dm}(0)}\right).$$

The proof is then analogous to the proof of proposition 7 (see appendix B.1.8).

Finally, note that

$$rr_{fs} = -(1-\mu\bar{B})rr_s$$

is identical to (31). Thus, proposition 8 carries over.

B.3.7 Proof of proposition 16

See the proof of the general model with infinite horizon in section B.4.3. While this case is not strictly nested within that case, it is straightforward to construct an analogous proof.

B.4 Dynamic model: General framework

In this section, I extend the analysis of section 4 to an infinite-horizon setting with Calvo pricing. Section B.4.1 presents the formal setup in detail. Section B.4.2 presents the planning problem. Section B.4.3 describes the steady state of the dynamic model. Section B.4.4 derives a quadratic approximation of the objective function (up to the portfolio) with four loss terms: lack of insurance, output gaps, inflation, and savings distortions. Section B.4.5 derives a first-order approximation of the constraints. Section B.4.6 presents the approximate problem and discusses the additional constraints one needs to add to make the solution optimal from the "timeless perspective", as discussed in section A.1. Section B.4.7 solves for the optimal path of variables in expectation in the periods after the shock hits. Section B.4.8 solves for the optimal innovations in the period when a shock hits. Sections B.4.9 and B.4.10 describe the optimal realized returns the planner promises and the optimal transfer from the rest of the world, respectively. Section B.4.11 presents proposition 19 and lemma 11, which describe the optimal portfolio choice, extending proposition 10 and lemma 5 in the main text, respectively, to a dynamic setting. Section B.4.12 presents proposition 20, which extends proposition 11 when there is a single "endogenous" asset, i.e. an asset whose return depends on policy - see section B.4.6 for a formal definition. Section B.4.13 shows how to solve the model when there are more endogenous assets. Section B.4.14 presents proposition

21, which extends proposition 12 without any qualifications. Finally, section B.4.15 shows how to back out the optimal time-varying capital controls (savings taxes) and how to solve the problem if only time-invariant capital controls are available.

B.4.1 Set up and competitive equilibrium

Financial assets Home agents can trade $J+1 \leq K$ assets with the rest of the world. For ease of exposition, I assume that one of these assets, labeled asset 0, is a short risk-free asset in foreign-currency with yield $R^*(\xi_t)$. The remaining assets may be of two categories: nominal assets, $\Theta_j \in \mathcal{J}_N$, which have payoffs denominated in home currency, and real assets, $\Theta_j \in \mathcal{J}_R$, which have payoffs denominated in foreign currency (i.e., tradable units). That is, let

$$\mathcal{Y}_t = \left\{ C_{Tt}, C_{Nt}, L_t, E_t^{-1} P_{Nt}, E_t^{-1} W_t, E_t^{-1} \int_0^1 \Pi_{Nt}(i) di \right\}$$

denote the real value of aggregate equilibrium variables (i.e., in foreign currency). Real assets have foreign-currency payoffs given by $\tilde{X}(\mathcal{Y}_t; \xi_t)$, where $\tilde{X}(\cdot)$ is a positive function. For example, a claim on the tradable endowment would be $\tilde{X}(\mathcal{Y}_t; \xi_t) = Y_T(\xi_t)$. Nominal assets have home-currency payoffs given by $\tilde{X}(\mathcal{Y}_t; \xi_t)$, so that their foreign-currency payoff is $E_t^{-1}\tilde{X}(\mathcal{Y}_t; \xi_t)$. For example, a short home-currency bond would be $\tilde{X}(\mathcal{Y}_t, \xi_t) = 1$. Accordingly, the yields R_j for real and nominal assets are denominated in foreign and home currency, respectively. Note that the return of a real asset may still be endogenous to monetary policy. For example, claims on the dollar value of nontradable firms would be $\tilde{X}(\mathcal{Y}_t; \xi_t) = E_t^{-1} \int_0^1 \Pi_{Nt}(i) di$.

In addition, I assume an asset j bought at t pays a coupon $\delta_j \tilde{X}(\mathcal{Y}_t; \xi_t)$ at $t+1$ and $1-\delta_j$ units of the asset at $t+1$. Furthermore, each unit of asset j pays a convenience yield $\Psi_j(\xi_t)$, which is zero at the steady state, i.e. $\Psi_j(\bar{\xi}) = 0$ (the convenience yield of asset 0 is normalized to 0). In sum, the realized excess return with respect to the short risk-free asset is given by

$$rr_{jt+1} = R_{jt} \left\{ \Psi(\xi_t) + \tilde{X}(\mathcal{Y}_{t+1}; \xi_{t+1})\delta_j + (1-\delta_j) R_{jt+1}^{-1} \right\} - R^*(\xi_t) \text{ for } j \in \mathcal{J}_R \tag{179}$$

$$rr_{jt+1} = R_{jt} E_t E_{t+1}^{-1} \left\{ \Psi(\xi_t) + \tilde{X}(\mathcal{Y}_{t+1}; \xi_{t+1})\delta_j + (1-\delta_j) R_{jt+1}^{-1} \right\} - R^*(\xi_t) \text{ for } j \in \mathcal{J}_N. \tag{180}$$

Consumers Consumers solve

$$\max_{\{C_{Tt}, C_{Nt}, L_t, nfa_t, \Theta_t\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(C_{Tt}, C_{Nt}, L_t; \xi_t)$$

subject to

$$C_{Tt} + E_t^{-1} P_{Nt} C_{Nt} + R^*(\xi_t)^{-1} nfa_t (1+\tau_{0t}) + \sum_{j>0} \Theta_{jt}(\tau_{jt} - \tau_{0t}) = \tag{181}$$

$$Y_T(\xi_t) + E_t^{-1} W_t L_t + E_t^{-1} \int_0^1 \Pi_{Nt}(i) di + nfa_{t-1} + \sum_{j>0} rr_{jt}\Theta_{jt-1} + T_t.$$

where $\Pi_{Nt}(i)$ are profits from nontradable firm i and $nfa_t = R^*(\xi_t) \sum_{j=0}^{J} \Theta_{jt}$ is the country's net-foreign-asset position multiplied by the foreign-currency short interest rate (a convenient normalization). Note that agents take as given rr_{jt}. Optimization over labor and tradable and nontradable consumption yields

$$\frac{\partial U}{\partial C_N}(t) / \frac{\partial U}{\partial C_T}(t) = \frac{P_{Nt}}{E_t} \tag{182}$$

$$(-\frac{\partial U}{\partial L}(t)) / \frac{\partial U}{\partial C_T}(t) = \frac{W_t}{E_t} \tag{183}$$

As in the general static model of section 4, I assume is a C_N is a CES composite of a continuum of varieties $C_{Nt}(i)$ with elasticity of substitution η. Optimization across varieties gives rise to the standard CES demand,

$$C_{Nt}(i) = \left(\frac{P_{Nt}(i)}{P_{Nt}}\right)^{-\eta} C_{Nt}, \tag{184}$$

where P_{Nt} is the ideal price index of nontradable goods,

$$P_{Nt} = \left(\int_0^1 P_{Nt}(i)^{1-\eta} di\right)^{\frac{1}{1-\eta}}. \tag{185}$$

Asset optimization yields a no-arbitrage condition,

$$\beta \mathbb{E}_t rr_{jt+1} \frac{\partial U}{\partial C_T}(t+1) = \frac{\partial U}{\partial C_T}(t)(\tau_{jt} - \tau_{0t}) \tag{186}$$

and an Euler equation,

$$\beta R_t^* \mathbb{E}_t \frac{\partial U}{\partial C_T}(t+1) = (1+\tau_{0t}) \frac{\partial U}{\partial C_T}(t). \tag{187}$$

The country's assets satisfy a no-Ponzi condition,

$$\lim_{t\to\infty} \prod_{s=0}^{t} R_s^{*-1} nfa_t = 0 \ a.s. \tag{188}$$

Foreigners I assume there are two types of foreigners. First, there is a large set of unsophisticated investors that provide an infinitely-elastic supply of the short risk-free asset in foreign currency at rate R_t^*. Their consumption is unaffected by policy at home. Second, there is a finite set of measure m of sophisticated investors that are willing to trade any asset with the home country and can also trade the risk-free asset with the unsophisticated investors. Asset optimization by the sophisticated investors yields

$$\mathbb{E}_t \left[rr_{jt+1} \frac{dU^*}{dC^*}(t+1) \right] = 0. \tag{189}$$

for assets $j > 0$. Since they can save and borrow in the risk-free bond

$$\frac{dU^*}{dC^*}(t) = \beta^*(\xi_t) R^*(\xi_t) \mathbb{E}_t \frac{dU^*}{dC^*}(t+1). \tag{190}$$

I assume that, at the steady state $\beta = \beta^*$. Using asset market clearing, their budget constraint is

$$C_t^* + R^*(\xi_t)^{-1} nfa_t^* = Y^*(\xi_t) - m^{-1} \sum_{j>0} rr_{jt} \Theta_{jt-1} + nfa_{t-1}^*, \tag{191}$$

where nfa_t^* is the (normalized) net-foreign-asset position of the sophisticated investors, which also satisfies a no-Ponzi condition:

$$\lim_{t\to\infty}\prod_{s=0}^{t} R_s^{*-1} nfa_t^* = 0 \ a.s. \tag{192}$$

I assume that if $\{C_t^*\}_t = \{Y^*(\xi_t)\}_t$ foreigners are indifferent between lending and borrowing, i.e. equation (190) holds. Thus, given some path for output, interest rate shocks R_t^* are rationalized by β^* shocks. Note that, since there is an infinitely elastic supply of the foreign-currency bond by the unsophisticated investors, $nfa_t + nfa_t^* \neq 0$.

Intermediate good producers Firms have access to a neoclassical technology

$$C_{Nt}(i) = F(L_t(i);\xi_t). \tag{193}$$

Note that technology is identical across firms and there are no idiosyncratic technology shocks. Thus, from the perspective of production efficiency, all firms should produce the same amounts. A set ϕ of firms $i \in \{\text{fix}\}$ cannot reset their price:

$$P_{Nt}(i) = P_{Nt-1}(i) \text{ for } i \in \{\text{fix}\}. \tag{194}$$

A set of firms $1-\phi$ $i \in \{\text{flex}\}$ can reset their price. The probability of belonging to this set is i.i.d. over time and across firms (i.e., Calvo pricing). I assume that there is a labor subsidy τ_L that offsets firms' desired mark up, i.e. $1-\tau_L = \frac{\eta-1}{\eta}$. Optimality gives rise to the condition

$$\mathbb{E}_t \sum_{s=0}^{\infty} (\beta\phi)^s \Lambda_{t+s} \left\{ P_{Nt}(i) - \frac{1}{\frac{\partial F}{\partial L}(L_{t+s}(i);\xi_{t+s})} W_{t+s} \right\} \left(\frac{P_{Nt}(i)}{P_{Nt+s}} \right)^{-\eta} C_{Nt+s} = 0 \text{ for } i \in \{\text{flex}\}, \tag{195}$$

where Λ_{t+s} is the firms' stochastic discount factor.[53]

Taxes The central government rebates the proceeds of the financial taxes $\{\tau_j\}$ lump-sum and the cost of the labor subsidy τ_L,

$$T_t = \tau_{0t} R^*(\xi_t)^{-1} nfa_t + \sum_{j>0} (\tau_{jt} - \tau_{0t})\Theta_{jt} - \tau_L W_t L_t.$$

Goods and labor market clearing Replacing firms' profits and taxes into (181), and using nontradable market clearing yields the country's budget constraint:

$$C_{Tt} + R^*(\xi_t)^{-1} nfa_t = Y_T(\xi_t) + nfa_{t-1} + \sum_{j>0} rr_{jt}\Theta_{jt-1}. \tag{196}$$

The market-clearing condition for labor is given by

$$L_t = \int_0^1 L_t(i)di. \tag{197}$$

[53]It would be natural to assume that this discount factor is that of home households. In any event, since the deterministic steady state is efficient, the choice of Λ_{t+s} is inconsequential in the approximate model.

Shocks Shocks follow a first-order Markov process,

$$\xi_t = V_\xi \xi_{t-1} + \epsilon_t,$$

where V_ξ is a matrix with eigenvalues of absolute value strictly less than one and ϵ_t is a compact-valued random variable.

Competitive equilibrium Next, I formally define a competitive equilibrium in this economy.

Definition 5. Given a Central Bank policy $(\{E_t\}_t,\{\tau_{jt}\}_{jt})$, an allocation $(\{C_{Tt}\}_t,\{C_{Nt}\}_t,\{L_t\}_t,\{nfa_t\}_t,$ $\{C_{Nt}(i)\}_{i,t},\{L_t(i)\},\{C^*_{Tt}\}_t,\{nfa^*_t\}_t,\{\Theta_{jt}\}_{j>0,t})$ together with prices $(\{P_{Nt}\}_t,\{W_t\},\{P_{Nt}(i)\}_{i,t},\{R_{jt}\}_{j>0,t})$ is a ***competitive equilibrium*** if and only if they solve (182)−(197).

B.4.2 Planning problem

The planning problem is to choose $(\{E_t\}_t, \{\tau_{jt}\}_{jt})$, an allocation, and prices to maximize

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \{U(C_{Tt}, C_{Nt}, L_t; \xi_t) + m\bar{\lambda} U^*(C^*_{Tt}; \xi_t)\}$$

subject to (182)−(197). In section B.4.6 below, I add "promise-keeping" constraints to the problem so that the resulting policy is optimal from the "timeless" perspective in the sense of Benigno and Woodford (2012). To arrive at a recursive formulation of the problem, it will prove convenient to define the value function,

$$\mathcal{V}_{t_0} = \sum_{t=t_0}^{\infty} \beta^{t-t_0} \{U(C_{Tt}, C_{Nt}, L_t; \xi_t) + m\bar{\lambda} U^*(C^*_{Tt}; \xi_t)\}.$$

B.4.3 Steady state

I study the solution around a deterministic steady state, which solves the planning problem in the absence of shocks once I add the relevant promise-keeping constraints in section B.4.6.[54] At the steady state, $R^* = \beta^{-1}$. Tradable consumption consumption is given by

$$\bar{C}_T = \bar{Y}_T + (1-\beta) n\bar{f}a.$$

I assume $n\bar{f}a^* = 0$ so $\bar{C}^* = \bar{Y}^*$. The steady-state price of asset j is given by

$$\bar{R}_j = \frac{1-\beta(1-\delta_j)}{\beta \delta_j \tilde{X}(\bar{\mathcal{Y}}, \bar{\xi})}.$$

The remaining steady-state variables $\left(\bar{C}_N, \bar{L}, \frac{\bar{P}_N}{\bar{E}}, \frac{\bar{W}}{\bar{P}_N}\right)$ solve the same equations as in appendix B.2.3.

[54]Without these constraints, the solution would not be stationary: In the first period, the planner could "surprise" markets and engineer realized returns in favor of the home country (if $\bar{\lambda} = 0$). The models in Chang and Velasco (2006) and Du, Pflueger and Schreger (2020) feature this mechanism. To study the problem without the initial promise-keeping constraints, one would first need to derive the appropriate non-stationary solution in the model without shocks and then approximate around this point. This would mostly affect the results in the first period; the analysis for subsequent periods would be essentially the same as the one I characterize in this appendix.

B.4.4 Approximate problem: Four loss terms

In this section, I derive an approximation to the objective function with four loss terms: deviations of the transfers from the first-best, deviations of savings from the first-best given transfers (henceforth, "savings gap"), output gaps, and inflation.

Lemma 9. *Around the deterministic steady state, the planner's objective function satisfies*

$$\mathcal{V}_{t_0} = \mathcal{A}_{t_0} - \frac{1}{2}\mathbb{E}_{t_0}\sum_{t=t_0}\beta^{t-t_0}\Bigg\{ \left(A_{\mathcal{T}\mathcal{T}} + \left(\frac{1-\beta}{m}\right)\gamma^*\left(\frac{\partial U}{\partial C_T} - \frac{\partial U^*}{\partial C_T}\bar{\lambda}\right)\right)(\mathcal{T}_t - \mathcal{T}_{cm,t})^2 \tag{198}$$
$$+ \lambda_x x_t^2 + \lambda_\pi \pi_{Nt}^2 + \frac{\partial U}{\partial C_T}\gamma_T\left(n\tilde{f}a_{t-1} - \beta n\tilde{f}a_t\right)^2\Bigg\} + t.i.p. + \mathcal{O}(\epsilon^3)$$

where $\mathcal{T}_t = \sum_{j>0}\bar{\Theta}_j rr_{jt}$ *is the transfer received by the home country from abroad at* t, $\mathcal{T}_{cm,t}$ *are the transfers the planner would choose under complete markets and flexible prices,*

$$\mathcal{T}_{cm,t} = \left(A_{\mathcal{T}\mathcal{T}} + \left(\frac{1-\beta}{m}\right)\gamma^*\left(\frac{\partial U}{\partial C_T} - \frac{\partial U^*}{\partial C_T}\bar{\lambda}\right)\right)^{-1} A_{\mathcal{T}\xi}\epsilon_t, \tag{199}$$

x_t *is the output gap,* π_{Nt} *is non-tradable price inflation, and* $n\tilde{f}a_t$ *is the "savings gap", i.e. the difference between the actual nfa position* nfa_t *and the one that the planner would choose given some exogenous transfers* $\{\mathcal{T}_t\}$, nfa_t^{fb}, *which satisfies*

$$nfa_t^{fb} = nfa_{t-1}^{fb} + \mathcal{T}_t + \tilde{k}_\xi \xi_t, \tag{200}$$

and $\mathcal{A}_{t_0}$ *collects terms that, given a* $\mathcal{T}_{t_0}$ *promise at* $t_0 - 1$, *are taken as given at* t_0,

$$\mathcal{A}_{t_0} = \mathcal{A}_{Ht_0} + \mathcal{A}_{Ft_0} + \left(\frac{\partial U}{\partial C_T} - \bar{\lambda}\frac{\partial U^*}{\partial C_T}\right)\left(\mathcal{T}_{t_0} - \gamma^*\mathcal{T}_{t_0}D_\xi Y^*\cdot\xi_{t_0} + \gamma^*\frac{1}{m}(1-\beta)\mathcal{T}_{t_0}^2\right).$$

The remaining constants are given by

$$A_{\mathcal{T}\mathcal{T}} = (1-\beta)\frac{\partial U}{\partial C_T}\left(\gamma_T + \gamma^* \frac{1}{m}\right) > 0 \tag{201}$$

$$A_{\mathcal{T}\xi} = \frac{\partial U}{\partial C_T}\left(\gamma^* D_\xi Y^* + \beta\gamma_T \tilde{k}_\xi + k_\xi\right) \tag{202}$$

$$\tilde{k}_\xi = \gamma_T^{-1}\left(k_\xi V_\xi - k_{L\xi}\right)\left(I - \beta V_\xi\right)^{-1} \tag{203}$$

$$k_\xi = -\gamma_T\left(D_\xi Y_T + \beta n\bar{f}a D_\xi \ln R^*\right) + \left(\frac{\partial U}{\partial C_T}\right)^{-1}\left(A_{c\xi} + A_{ll}^{-1}A_{cl}A_{l\xi}\right). \tag{204}$$

$$k_{L\xi} = k_\xi - D_\xi \ln R^* \tag{205}$$

$$\lambda_\pi = \frac{\phi}{(1-\beta\phi)(1-\phi)}A_{pp} > 0 \tag{206}$$

$$\begin{aligned}\mathcal{A}_{Ht_0} = {} & \frac{\partial U}{\partial C_T} nfa_{t_0-1}\left\{1 - (1-\beta)\gamma_T \mathcal{T}_{t_0} + \left(k_\xi + \beta\gamma_T\tilde{k}_\xi\right)\cdot \xi_{t_0}\right\} \\ & - \frac{1}{2}\frac{\partial U}{\partial C_T}(1-\beta)\gamma_T\left(nfa_{t_0-1}^2 - n\tilde{f}a_{t_0-1}^2\right)\end{aligned} \tag{207}$$

$$\mathcal{A}_{Ft_0} = \frac{dU^*}{dC^*} nfa^*_{t_0-1}\left(1 + (1-\beta)\gamma^* m^{-1}\mathcal{T}_{t_0} - \gamma^* D_\xi Y^* \cdot \xi_{t_0}\right) - \frac{1}{2}\frac{dU^*}{dC^*}(1-\beta)\gamma^* m^{-1}\left(nfa^*_{t_0-1}\right)^2 \tag{208}$$

The constants A_{pp}, A_{cl}, $A_{c\xi}$, A_{ll}, $A_{l\xi}$, λ_x, γ_T and γ^* are still given by (118), (119), (120), (121), (122), (130), (132) and (133), respectively. Note that a global planner ($\frac{\frac{\partial U}{\partial C_T}}{\frac{dU^*}{dC^*}} = \bar{\lambda}$) would put a smaller weight on the insurance term and choose larger transfers under complete markets than a home planner ($\bar{\lambda} = 0$).

First, note that the approximation to flow utility given by lemma 6 is still valid. Adding (117) over time and taking expectations at $t = t_0$:

$$\begin{aligned}\mathcal{U}_{t_0} = \mathbb{E}_{t_0}\sum_{t=t_0}^{\infty}\beta^{t-t_0}\Bigg\{ & \frac{\partial U}{\partial C_T}\bar{C}_T\left(c_{Tt} + \frac{1}{2}c_{Tt}^2\right) - \frac{1}{2}A_{pp}\left(\int_0^1 (p_{Nt}(i) - p_{Nt})^2 di\right) \\ & + \frac{1}{2}\left(\frac{\partial^2 U}{\partial C_T^2} + \frac{A_{cl}^2}{A_{ll}}\right)\bar{C}_T^2 c_{Tt}^2 + \bar{C}_T c_{Tt}\left(A_{c\xi} + \frac{A_{cl}}{A_{ll}}A_{l\xi}\right)\cdot\xi_t - \frac{1}{2}\lambda_x x_t^2\Bigg\} + \text{t.i.p.} + \mathcal{O}(\epsilon^3),\end{aligned} \tag{209}$$

where, as in equation (136), I have rewritten flow utility in terms of the output gap. Using a second-order approximation of the country's budget constraint (196), adding over time, using the no-Ponzi condition (188), and taking expectations at $t = t_0$, yields

$$\mathbb{E}_{t_0}\sum_{t=t_0}^{\infty}\beta^{t-t_0}\bar{C}_T\left(c_t + \frac{1}{2}c_t^2\right) = nfa_{t_0-1} + \mathbb{E}_{t_0}\sum_{t=t_0}^{\infty}\beta^{t-t_0}\left(\mathcal{T}_t + \beta nfa_t D_\xi \ln R^* \cdot \xi_t\right) + \mathcal{O}(\epsilon^3) \tag{210}$$

In addition, I use the following result from Woodford (2003),

$$\sum_{t=0}^{\infty}\beta^t\left(\int_0^1 (p_{Nt}(i) - p_{Nt})^2 di\right) = \frac{\phi}{(1-\beta\phi)(1-\phi)}\sum_{t=0}^{\infty}\beta^t \pi_{Nt}^2. \tag{211}$$

Using (210) and (211), one may rewrite (209),

$$\mathcal{U}_{t_0} = \frac{\partial U}{\partial C_T} nfa_{t_0-1} + \mathbb{E}_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left\{ \frac{\partial U}{\partial C_T} \left(\mathcal{T}_t + \beta nfa_t D_\xi \ln R^* \cdot \xi_t\right) - \frac{1}{2}\lambda_\pi \pi_{Nt}^2 \right. \tag{212}$$

$$\left. + \frac{1}{2}\left(\frac{\partial^2 U}{\partial C_T^2} + \frac{A_{cl}^2}{A_{ll}}\right) \bar{C}_T^2 c_{Tt}^2 + \bar{C}_T c_{Tt} \left(A_{c\xi} + \frac{A_{cl}}{A_{ll}} A_{l\xi}\right) \cdot \xi_t - \frac{1}{2}\lambda_x x_t^2 \right\} + \text{t.i.p.} + \mathcal{O}(\epsilon^3).$$

Next, I use a first-order approximation of the budget constraint,

$$\bar{C}_T c_{Tt} + \beta nfa_t - \beta n\bar{f}a D_\xi \ln R^* \cdot \xi_t = D_\xi Y_T \cdot \xi_t + \mathcal{T}_t + nfa_{t-1} + \mathcal{O}(\epsilon^2),$$

to rewrite (212) as

$$\mathcal{U}_{t_0} = \frac{\partial U}{\partial C_T} nfa_{t_0-1} \left\{1 - \gamma_T \mathcal{T}_{t_0} + k_\xi \cdot \xi_{t_0}\right\} \tag{213}$$

$$+ \mathbb{E}_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left\{ \frac{\partial U}{\partial C_T} \left(\mathcal{T}_t - \frac{1}{2}\gamma_T \mathcal{T}_t^2 - \frac{1}{2}\gamma_T \left(nfa_{t-1} - \beta nfa_t\right)^2 + \beta \gamma_T nfa_t \mathcal{T}_t \right.\right.$$

$$\left.\left. + \mathcal{T}_t k_\xi \xi_t + \beta \left(k_\xi \cdot \xi_{t+1} - k_{L\xi} \cdot \xi_t\right) nfa_t \right) - \frac{1}{2}\lambda_\pi \pi_{Nt}^2 - \frac{1}{2}\lambda_x x_t^2 \right\} + \text{t.i.p.} + \mathcal{O}(\epsilon^3).$$

Next, I use the following lemma to define the "savings gap": the deviation of the actual savings rate nfa_t from the one that would emerge if transfers were independent of the monetary and savings policy, nfa_t^{fb}.

Lemma 10. *Given some exogenous path for transfers* $\{\mathcal{T}_t\}_{t=t_0}^{\infty}$ *and initial foreign position* $nfa_{t_0-1}^{fb}$*, maximizing (213) without constraints yields* $x_t = \pi_{Nt} = 0 \ \forall t$ *and (200).*

Proof. The FOC of (213) with respect to nfa_t yields

$$-\beta \mathbb{E}_t nfa_{t+1}^{\text{fb}} + (1+\beta) nfa_t^{\text{fb}} - nfa_{t-1}^{\text{fb}} = \mathcal{T}_t - \gamma_T^{-1} k_{L\xi} \cdot \xi_t + \gamma_T^{-1} k_\xi \cdot \mathbb{E}_t \xi_{t+1}.$$

Using the law of iterated expectations for $s > 0$,

$$-\beta \mathbb{E}_t nfa_{t+s+1}^{\text{fb}} + (1+\beta) \mathbb{E}_t nfa_{t+s}^{\text{fb}} - \mathbb{E}_t nfa_{t+s-1}^{\text{fb}} = -\gamma_T^{-1} k_{L\xi} \cdot \mathbb{E}_t \xi_{t+s} + \gamma_T^{-1} k_\xi \cdot \mathbb{E}_t \xi_{t+s+1}.$$

This can be written as a system of difference equations with a root outside the unit circle, a unit root, and K stationary roots corresponding to the shocks. Picking the initial condition nfa_t appropriately so that the system does not diverge, I obtain (200). □

After some algebra, one may rewrite (213) in terms of "gaps", $n\tilde{f}a_t = nfa_t - nfa_t^{\text{fb}}$,

$$\mathcal{U}_{t_0} = \mathcal{A}_{Ht_0} + \mathbb{E}_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left\{ \frac{\partial U}{\partial C_T} \left(\mathcal{T}_t - \frac{1}{2}(1-\beta)\gamma_T \mathcal{T}_t^2 - \frac{1}{2}\gamma_T \left(n\tilde{f}a_{t-1} - \beta n\tilde{f}a_t\right)^2 \right.\right. \tag{214}$$

$$\left.\left. + \mathcal{T}_t \left(k_\xi + \beta \gamma_T \tilde{k}_\xi\right) \xi_t \right) - \frac{1}{2}\lambda_\pi \pi_{Nt}^2 - \frac{1}{2}\lambda_x x_t^2 \right\} + \text{t.i.p.} + \mathcal{O}(\epsilon^3).$$

where $\mathcal{A}_{Ht_0}$ is given by (207).

Next, I approximate foreign utility, $\mathcal{U}^*_{t_0} = \sum_{t=t_0}^{\infty} \prod_{s=t_0}^{t-1} \beta^*(\xi_s) U(C^*_{Tt})$,[55]

$$\mathcal{U}^*_{t_0} = \mathbb{E}_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \Bigg\{ \frac{dU^*}{dC^*} \bar{C}^* (c^*_t + \frac{1}{2} c^{*2}_t) + \frac{d^2 U^*}{dC^{*2}} C^{*2} c^{*2}_{Tt} \quad (215)$$

$$+ \beta \left(D_\xi \ln \beta^* \cdot \xi_t \right) \left(\sum_{s=0}^{\infty} \beta^s \frac{dU^*}{dC^*} \bar{C}^* c^*_{t+1+s} \right) \Bigg\} + \text{t.i.p.} + \mathcal{O}(\epsilon^3).$$

A second-order approximation of the foreign budget constraint yields

$$\mathbb{E}_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left(\bar{C}^* c^*_t + \frac{1}{2} \bar{C}^* c^{*2}_t \right) = nfa^*_{t_0-1} + \mathbb{E}_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left(-m^{-1} \mathcal{T}_t + \beta nfa^*_t D_\xi \ln R^* \cdot \xi_t \right) + \text{t.i.p.} + \mathcal{O}(\epsilon^3) \quad (216)$$

Using (216), one may rewrite (215),

$$\mathcal{U}^*_{t_0} = \frac{dU^*}{dC^*} nfa^*_{t_0-1} + \mathbb{E}_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \Bigg\{ - \frac{dU^*}{dC^*} m^{-1} \mathcal{T}_t + \frac{dU^*}{dC^*} \beta nfa^*_t D_\xi \ln R^* \cdot \xi_t \quad (217)$$

$$+ \frac{d^2 U^*}{dC^{*2}} C^{*2} c^{*2}_{Tt} + \beta \left(D_\xi \ln \beta^* \cdot \xi_t \right) \left(\sum_{s=0}^{\infty} \beta^s \frac{dU^*}{dC^*} \bar{C}^* c^*_{t+1+s} \right) \Bigg\} + \text{t.i.p.} + \mathcal{O}(\epsilon^3).$$

Next, I use a first-order approximation of the foreign budget constraint,[56]

$$\bar{C}^* c^*_t + \beta nfa^*_t = D_\xi Y^* \cdot \xi_t - m^{-1} \mathcal{T}_t + nfa^*_{t-1} + \mathcal{O}(\epsilon^2)$$

and, iterating forward this equation and using that $\mathbb{E}_{t-1} \mathcal{T}_t = \mathcal{O}(\epsilon^2)$,

$$\mathbb{E}_t \left(\sum_{s=0}^{\infty} \beta^s \bar{C}^* c^*_{t+1+s} \right) = nfa_t + \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \left(D_\xi Y^* \cdot \xi_{t+1+s} \right) + \mathcal{O}(\epsilon^3),$$

to rewrite (217) as follows,

$$\mathcal{U}^*_{t_0} = \frac{dU^*}{dC^*} nfa^*_{t_0-1} \left(1 + m^{-1} \gamma^* \mathcal{T}_{t_0} \right) + \mathbb{E}_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \frac{dU^*}{dC^*} \Bigg\{ - m^{-1} \mathcal{T}_t \quad (218)$$

$$+ \beta nfa^*_t \left(D_\xi \ln R^* + D_\xi \beta^* \right) \cdot \xi_t - \frac{1}{2} \gamma^* \left(nfa^*_{t-1} - \beta nfa^*_t \right)^2 - \gamma^* \beta m^{-1} \mathcal{T}_t nfa^*_t$$

$$- \gamma^* \left(nfa^*_{t-1} - \beta nfa^*_t \right) D_\xi Y^* \cdot \xi_t - \frac{1}{2} \gamma^* m^{-2} \mathcal{T}^2_t + \gamma^* m^{-1} \mathcal{T}_t D_\xi Y^* \cdot \xi_t \Bigg\} + \text{t.i.p.} + \mathcal{O}(\epsilon^3).$$

A first-order approximation to the foreign Euler equation yields

$$\gamma^* \bar{C}^* c^*_t = - \left(D_\xi \ln R^* + D_\xi \beta^* \right) \cdot \xi_t + \gamma^* \bar{C}^* \mathbb{E}_t c^*_{t+1} + \mathcal{O}(\epsilon^2)$$

[55] I use the convention that $\prod_{t_0}^{t_0-1} \beta^*(\xi_s) = 1$.
[56] Recall that I assumed $\bar{nfa}^* = 0$.

Since R^* is consistent with $y_t = c_t^* \ \forall t$ if $\mathcal{T}_t = 0 \ \forall t$,

$$\gamma^* D_\xi Y^* \cdot \xi_t = -\left(D_\xi \ln R^* + D_\xi \beta^*\right) \cdot \xi_t + \gamma^* D_\xi Y^* \cdot \mathbb{E}_t \xi_{t+1} + \mathcal{O}(\epsilon^2) \tag{219}$$

Thus, for all $s \geq 0$,

$$\bar{C}^* c_t^* - D_\xi Y^* \cdot \xi_t = \bar{C}^* \mathbb{E}_t c_{t+s+1}^* - D_\xi Y^* \cdot \mathbb{E}_t \xi_{t+s+1} + \mathcal{O}(\epsilon^2).$$

Then, using the budget constraint,

$$nfa_t^* = -m^{-1}\mathcal{T}_t + nfa_{t-1}^* + \mathcal{O}(\epsilon^2)$$

Replacing back in (218) and using $\mathbb{E}_{t-1}\mathcal{T}_t = \mathcal{O}(\epsilon^2)$ and (219) yields, after some algebra,

$$\mathcal{U}_{t_0}^* = \mathcal{A}_{Ft_0} + \mathbb{E}_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \frac{dU^*}{dC^*} \left(-m^{-1}\mathcal{T}_t + m^{-1}\mathcal{T}_t \gamma^* D_\xi Y^* \cdot \xi_t - \frac{1}{2} m^{-2}(1-\beta)\gamma^* \mathcal{T}_t^2\right) + \text{t.i.p.} + \mathcal{O}(\epsilon^3) \tag{220}$$

where $\mathcal{A}_{Ft_0}$ is given by (208).

A second-order approximation to the foreign no-arbitrage condition yields, for $t > t_0$,

$$\mathbb{E}_{t-1}\mathcal{T}_t - \gamma^* \mathbb{E}_{t-1}\mathcal{T}_t \left(D_\xi Y^* \cdot \xi_t - (1-\beta) m^{-1} \mathcal{T}_t\right) = \mathcal{O}(\epsilon^3) \tag{221}$$

Using (221) to replace the linear term in transfers for $t > t_0$ in (214) and (220), and noting that $\mathcal{V}_{t_0} = \mathcal{U}_{t_0} + m\bar{\lambda}\mathcal{U}_{t_0}^*$,

$$\begin{aligned}\mathcal{V}_{t_0} = \mathcal{A}_{t_0} + \frac{1}{2}\mathbb{E}_{t_0} \sum_{t=t_0} \beta^t \Bigg\{ &- \left(A_{\mathcal{T}\mathcal{T}} + \left(\frac{1-\beta}{m}\right)\gamma^* \left(\frac{\partial U}{\partial C_T} - \frac{dU^*}{dC^*}\bar{\lambda}\right)\right)\mathcal{T}_t^2 \\ &+ \mathcal{T}_t A_{\mathcal{T}\xi} \cdot \xi_t - \lambda_x x_t^2 - \lambda_\pi \pi_{Nt}^2 - \frac{\partial U}{\partial C_T}\gamma_T \left(n\tilde{f}a_{t-1} - \beta n\tilde{f}a_t\right)^2 \Bigg\} + \text{t.i.p.} + \mathcal{O}(\epsilon^3).\end{aligned} \tag{222}$$

Note that, at $t = t_0$, there is no analogue of (221) so $\mathcal{A}_{t_0}$ includes terms with $\mathcal{T}_{t_0}$. If markets are complete, then the planner can attain the first best, eliminating inflation, savings and output gaps. Setting $x_t = \pi_{Nt} = n\tilde{f}a_t = 0 \ \forall t$, and maximizing (222) with respect to $\mathcal{T}_t$ subject to $\mathbb{E}_{t-1}\mathcal{T}_t = 0$ for $t > t_0$ yields (199).[57] Replacing back in (222) yields (198).

B.4.5 Approximate constraints

In this section, I derive an approximation to the constraints of the problem: the Phillips curve - a log-linearized version of (195), and the law of motion of asset prices and transfers as a function of realized excess returns.

Phillips curve Following the usual steps, one can show that a log-linear expansion of (195) yields

$$\pi_{Nt} = \kappa x_t + \beta \mathbb{E}_t \pi_{Nt+1},$$

[57] At $t = t_0$, $\mathcal{T}_{cm,t_0}$ is the one that would have been chosen at $t = t_0 - 1$ if the initial period had been $t = t_0 - 1$.

where

$$\kappa = \left(\frac{1-\phi}{\phi}\right)\left(\frac{1-\beta\phi}{1-\eta\left(\frac{\partial F}{\partial L}\right)^{-2}\frac{\partial^2 F}{\partial L^2}\bar{C}_N}\right)\bar{C}_N^{-1}(\frac{\partial U}{\partial C_N})^{-1}\lambda_x.$$

Asset prices First, note that using equilibrium relationships, one can rewrite the payoff function as a function of tradable consumption at t, C_{Tt}, the output gap, x_t, and output and price dispersion, Δ and Δ^p, respectively:

$$X_j(x_t, C_{Tt}, \Delta_t, \Delta_t^p; \xi_t) = \tilde{X}_j\left(\mathcal{Y}_t(x_t, C_{Tt}, \Delta_t, \Delta_t^p; \xi_t); \xi_t\right)$$

To see this, note that L_t is by definition related to the output gap and flexible-price labor

$$F(L_t; \xi_t) = \bar{C}_N x_t + F(L_t^{\text{flex}}(C_{Tt}, \xi_t); \xi_t),$$

C_{Nt} is related to labor and output dispersion Δ,

$$C_{Nt} = \Delta_t^{-1} F(L_t; \xi_t),$$

P_{Nt}/E_t and W_t/E_t follow from the FOCs (182) and (183), and the nontradable mutual fund return satisfies,

$$E_t^{-1}\int_0^1 \Pi_{Nt}(i)di = E_t^{-1}P_{Nt}\Delta_t^p C_{Nt} - E_t^{-1}W_t L_t$$

where

$$\Delta_t^p = \int_0^1 \left(\frac{P_{Nt}(i)}{P_{Nt}}\right)^{1-\eta} di.$$

Since Δ_t and Δ_t^p are zero to first order,

$$X_t = \frac{\partial X}{\partial x}x_t + \frac{\partial X}{\partial C_T}\bar{C}c_{Tt} + D_\xi X \cdot \xi_t + \mathcal{O}(\epsilon^2).$$

A first-order approximation of (179) and (180) yields

$$\begin{aligned}\beta rr_{jt+1} = {} & (1-\beta(1-\delta_j))\left(\frac{D_\xi \Psi_j}{\bar{X}\delta_j}\cdot\xi_t + \frac{\partial \ln X}{\partial C_T}\bar{C}_T c_{T+1} + \frac{\partial \ln X}{\partial x}x_{t+1} + D_\xi \ln X \cdot \xi_{t+1}\right) \\ & - \beta(1-\delta_j)r_{jt+1} - \mathbf{1}_{j\in\mathcal{J}_N}\Delta e_{t+1} - D_\xi \ln R^* \cdot \xi_t + r_{jt} + \mathcal{O}(\epsilon^2),\end{aligned} \tag{223}$$

where $\mathbf{1}_{j\in\mathcal{J}_N}$ is an indicator that takes a value of one if asset j is a nominal asset.

Next, note that equation (146) is still valid, i.e.

$$e_t = p_{Nt} + k_{ec}\bar{C}_T c_{Tt} + k_{e\xi}\cdot\xi_t + k_{ex}x_t + \mathcal{O}(\epsilon^2). \tag{224}$$

Using this and a first-order approximation to the budget constraint to substitute out $\bar{C}_T c_{Tt}$, (223) becomes, after some algebra,

$$\begin{aligned}\beta rr_{jt+1} = {} & -\beta(1-\delta_j)r_{jt+1} + r_{jt} + k^1_{r_j L\xi}\cdot\xi_t + k^1_{r_j\xi}\cdot\xi_{t+1} \\ & + (1-\beta(1-\delta_j))\left(\frac{\partial \ln X}{\partial C_T}\left(\mathcal{T}_{t+1} + nfa_t - \beta nfa_{t+1}\right) + \frac{\partial \ln X}{\partial x}x_{t+1}\right) \\ & - \mathbf{1}_{j\in\mathcal{J}_N}\left(\pi_{Nt+1} + k_{ec}\left((\mathcal{T}_{t+1} - \mathcal{T}_t) + (1+\beta)nfa_t - \beta nfa_{t+1} - nfa_{t-1}\right) + k_{ex}\Delta x_{t+1}\right)\end{aligned}$$

where

$$k^1_{r_jL\xi} = (1-\beta(1-\delta_j))\left(\frac{D_\xi \Psi_j}{\bar{X}\delta_j}\right) - D_\xi \ln R^* + \mathbf{1}_{j\in\mathcal{J}_N}\left(k_{ec}D_\xi Y_T + k_{ec}\beta n\bar{f}aD_\xi \ln R^* + k_{e\xi}\right)$$
$$k^1_{r_j\xi} = (1-\beta(1-\delta_j))\left(D_\xi \ln X + \frac{\partial \ln X}{\partial C_T}D_\xi Y_T + \frac{\partial \ln X}{\partial C_T}\beta n\bar{f}aD_\xi \ln R^*\right)$$
$$- \mathbf{1}_{j\in\mathcal{J}_N}\left(k_{ec}D_\xi Y_T + k_{ec}\beta n\bar{f}aD_\xi \ln R^* + k_{e\xi}\right).$$

Rewriting this in terms of gaps $n\tilde{f}a_t$,

$$\begin{aligned}\beta rr_{jt+1} &= -\beta(1-\delta_j)r_{jt+1} + r_{jt} + k_{r_jL\xi}\cdot\xi_t + k_{r_j\xi}\cdot\xi_{t+1} \\ &+ (1-\beta(1-\delta_j))\left(\frac{\partial \ln X}{\partial C_T}\left((1-\beta)\left(nfa^{\text{fb}}_t + \mathcal{T}_{t+1}\right) + n\tilde{f}a_t - \beta n\tilde{f}a_{t+1}\right) + \frac{\partial \ln X}{\partial x}x_{t+1}\right) \\ &- \mathbf{1}_{j\in\mathcal{J}_N}\left(\pi_{Nt+1} + k_{ec}\left((1-\beta)\mathcal{T}_{t+1} + (1+\beta)n\tilde{f}a_t - \beta n\tilde{f}a_{t+1} - n\tilde{f}a_{t-1}\right) + k_{ex}\Delta x_{t+1}\right)\end{aligned} \tag{225}$$

where

$$k_{r_jL\xi} = k^1_{r_jL\xi} - k_{ec}\mathbf{1}_{j\in\mathcal{J}_N}\tilde{k}_\xi$$
$$k_{r_j\xi} = k^1_{r_j\xi} - \beta\left(1-\beta(1-\delta_j)\right)\frac{\partial \ln X}{\partial C_T}\tilde{k}_\xi + \beta k_{ec}\mathbf{1}_{j\in\mathcal{J}_N}\tilde{k}_\xi.$$

Next, use that to first order $\mathbb{E}_t rr_{jt+1}$ is zero to solve for the yield r_{jt},

$$\begin{aligned}r_{jt} &= \beta(1-\delta_j)\mathbb{E}_t r_{jt+1} - k_{r_jL\xi}\cdot\xi_t - k_{r_j\xi}\cdot\mathbb{E}_t\xi_{t+1} \\ &- (1-\beta(1-\delta_j))\left(\frac{\partial \ln X}{\partial C_T}\left((1-\beta)nfa^{\text{fb}}_t + n\tilde{f}a_t - \beta\mathbb{E}_t n\tilde{f}a_{t+1}\right) + \frac{\partial \ln X}{\partial x}\mathbb{E}_t x_{t+1}\right) \\ &+ \mathbf{1}_{j\in\mathcal{J}_N}\left(\mathbb{E}_t\pi_{Nt+1} + k_{ec}\left((1+\beta)n\tilde{f}a_t - \beta\mathbb{E}_t n\tilde{f}a_{t+1} - n\tilde{f}a_{t-1}\right) + k_{ex}\mathbb{E}_t\Delta x_{t+1}\right).\end{aligned} \tag{226}$$

Next, I use that iterating backwards, $nfa^{\text{fb}}_t = \sum_{s=-\infty}^{t}\mathcal{T}_s + nfa^{\text{fb}}_t(0)$ where $nfa^{\text{fb}}_t(0)$ is the net-foreign asset position consistent with $\mathcal{T}_s = 0\ \forall s \le t$. Then, I define the associated no-gaps-no-transfers yield $r_{jt}(0)$,

$$r_{jt}(0) = \beta(1-\delta_j)\mathbb{E}_t r_{jt+1}(0) - k_{r_jL\xi}\cdot\xi_t - k_{r_j\xi}\cdot\mathbb{E}_t\xi_{t+1} - (1-\beta(1-\delta_j))\left(\frac{\partial \ln X}{\partial C_T}(1-\beta)nfa^{\text{fb}}_t(0)\right).$$

Note that $r_{jt}(0)$ is independent of policy. Henceforth, it will prove convenient to define a "normalized" yield of asset j, $\tilde{r}_{jt}$, given by

$$\tilde{r}_{jt} = r_{jt} - r_{jt}(0) + \frac{\partial \ln X}{\partial C_T}(1-\beta)\left(n\tilde{f}a_t + \sum_{s=-\infty}^{t}\mathcal{T}_s\right) + \mathbf{1}_{j\in\mathcal{J}_N}\left(k_{ex}x_t - k_{ec}\Delta n\tilde{f}a_t\right).$$

After this normalization, (226) can be rewritten as

$$\tilde{r}_{jt} = \beta(1-\delta_j)\mathbb{E}_t\tilde{r}_{jt+1} - k_{rr_j\pi}\mathbb{E}_t\pi_{Nt+1} - k_{rr_jx}\mathbb{E}_t x_{t+1} - \beta k_{rr_jb}\mathbb{E}_t\Delta n\tilde{f}a_{t+1} \tag{227}$$

where

$$k_{rr_jb} \equiv \delta_j \left(k_{ec}\mathbf{1}_{j\in\mathcal{J}_N} - \frac{\partial \ln X}{\partial C} \right)$$
$$k_{rr_jx} \equiv (1-\beta(1-\delta_j)) \left(\frac{\partial \ln X}{\partial x} - k_{ex}\mathbf{1}_{j\in\mathcal{J}_N} \right)$$
$$k_{rr_j\pi} \equiv -\mathbf{1}_{j\in\mathcal{J}_N}.$$

Next, I define the no-gaps-no-transfers realized returns $rr_{jt}(0)$ analogously:

$$\begin{aligned}\beta rr_{jt+1}(0) &= -\beta(1-\delta_j) r_{jt+1}(0) + r_{jt}(0) + k_{r_jL\xi}\cdot\xi_t + k_{r_j\xi}\cdot\xi_{t+1} \\ &+ (1-\beta(1-\delta_j)) \left(\frac{\partial \ln X}{\partial C_T} (1-\beta) nfa_t^{\text{fb}}(0) \right).\end{aligned} \tag{228}$$

Subtracting (228) from (225) yields, using the definition of $\tilde{r}_{jt}$, (227),

$$\beta rr_{jt+1} - \beta rr_{jt+1}(0) = \tilde{r}_{jt} - \beta(1-\delta_j)\tilde{r}_{jt+1} + k_{rr_j\pi}\pi_{Nt+1} + k_{rr_jx}x_{t+1} + \beta k_{rr_jb}\Delta \tilde{nfa}_{t+1} + \beta\mu_j\mathcal{T}_{t+1} + \mathcal{O}(\epsilon^2) \tag{229}$$

where

$$\mu_j = \beta^{-1}(1-\beta)\left(\frac{\partial \ln X}{\partial C} - k_{ec}\mathbf{1}_{j\in\mathcal{J}_N} \right).$$

B.4.6 Approximate problem: Set up

In this section, I set up the approximate problem. I begin by discussing the problem from the perspective of time t_0. Then, I derive the set of "promise-keeping" constraints one needs to add to make the problem recursive, i.e. study the optimal policy from a "timeless" perspective, as discussed in section A.1. This justifies studying the problem around the deterministic steady state characterized in section B.4.3. Before doing so, I will sort assets into two classes: endogenous assets $\mathcal{J}_D$ and exogenous assets $\mathcal{J}_X$. Intuitively, endogenous assets are those whose returns can be affected by policy. Note that endogenous assets are not necessarily nominal. In general equilibrium, the return of a real asset can be endogenous to policy, e.g. claims on non-tradable firms.

Definition 6. An asset j is exogenous to policy, i.e. $j \in \mathcal{J}_X$ if $k_{rr_j\pi} = k_{rr_jb} = k_{rr_jx} = 0$. If an asset j is not exogenous to policy, then it is endogenous, i.e. $j \in \mathcal{J}_D$. The number of endogenous assets is $J_D = \#(\mathcal{J}_D)$.

The problem at $t = t_0$ is choosing $\{x_t, \pi_{Nt}, \tilde{nfa}_t, rr_t, \tilde{r}_t, \mathcal{T}_t, \bar{\Theta}_t\}$ to maximize

$$\begin{aligned}\mathcal{A}_{t_0} - \frac{1}{2}\mathbb{E}_{t_0}\sum_{t=t_0}\beta^{t-t_0}\Bigg\{ &\left(A_{\mathcal{T}\mathcal{T}} + \left(\frac{1-\beta}{m}\right)\left(\frac{\partial U}{\partial C_T} - \frac{\partial U^*}{\partial C_T}\bar{\lambda}\right)\gamma^* \right)(\mathcal{T}_t - \mathcal{T}_{cm,t})^2 \\ &+ \lambda_x x_t^2 + \lambda_\pi \pi_{Nt}^2 + \frac{\partial U}{\partial C_T}\gamma_T\left(\tilde{nfa}_{t-1} - \beta\tilde{nfa}_t\right)^2\Bigg\} + \text{t.i.p.} + \mathcal{O}(\epsilon^3)\end{aligned} \tag{230}$$

subject to

$$\mathbb{E}_t rr_{jt+1} = 0 \tag{231}$$
$$\pi_{Nt} = \kappa x_t + \beta \mathbb{E}_t \pi_{Nt+1} \tag{232}$$
$$\beta rr_{jt} - \beta rr_{jt}(0) = \tilde{r}_{jt-1} - \beta(1-\delta_j)\tilde{r}_{jt} + k_{rr_j\pi}\pi_{Nt} + k_{rr_jx}x_t + \beta k_{rr_jb}\Delta n\tilde{f}a_t + \beta\mu_j \mathcal{T}_t \tag{233}$$
$$\mathcal{T}_t = \sum_j rr_{jt}\bar{\Theta}_{jt-1} \tag{234}$$

$\forall t \geq t_0$ and $j \in \mathcal{J}_D$ with $n\tilde{f}a_{t_0-1}, \bar{\Theta}_{t_0-1}$ and $\tilde{r}_{jt_0-1}$ given. To make the problem recursive, I add two promise-keeping constraints at $t = t_0$ for each of the two forward-looking constraints (231) and (232). First, note that as long as there is an endogenous asset, i.e. $j \neq \emptyset$, a home planner would like to manipulate the initial returns of asset j, rr_{jt_0}, in its country's favor, i.e. to increase $\mathcal{T}_{t_0}$. Thus, in the initial period one needs to add promises

$$rr_{jt_0} = \bar{rr}_{jt_0}. \tag{235}$$

Note that this is really only necessary for endogenous assets, which are the ones that the planner can manipulate. The other promise is standard and relates to inflation,

$$\pi_{Nt_0} = \bar{\pi}_{Nt_0}. \tag{236}$$

Using promises (235)−(236), and defining $\boldsymbol{y}_t = \{\pi_{Nt}, rr_t, n\tilde{f}a_{t-1}, \tilde{r}_{jt-1}, \bar{\Theta}_{t-1}\}$, the problem can be written recursively as follows,

$$\mathcal{V}(y_t;\xi_t) = \max_{\{\boldsymbol{y}_{t+1},x_t,\mathcal{T}_{t+1},\bar{\Theta}_t\}} -\frac{1}{2}\mathbb{E}_t\Bigg\{\beta\left(A_{\mathcal{T}\mathcal{T}} + \left(\frac{1-\beta}{m}\right)(\frac{\partial U}{\partial C_T} - \frac{\partial U^*}{\partial C_T}\bar{\lambda})\gamma^*\right)(\mathcal{T}_{t+1} - \mathcal{T}_{cm,t+1})^2$$
$$+ \lambda_x x_t^2 + \beta\lambda_\pi \pi_{Nt+1}^2 + \frac{\partial U}{\partial C_T}\gamma_T\left(n\tilde{f}a_{t-1} - \beta n\tilde{f}a_t\right)^2 + \beta\mathcal{V}(y_{t+1};\xi_{t+1})\Bigg\} + \text{t.i.p.} + \mathcal{O}(\epsilon^3)$$

subject to

$$\pi_{Nt} = \kappa x_t + \beta \mathbb{E}_t \pi_{Nt+1}$$
$$\mathbb{E}_t rr_{jt+1} = 0,$$
$$\beta rr_{jt} - \beta rr_{jt}(0) = \tilde{r}_{jt-1} - \beta(1-\delta_j)\tilde{r}_{jt} + k_{rr_j\pi}\pi_{Nt} + k_{rr_jx}x_t + \beta k_{rr_jb}\Delta n\tilde{f}a_t + \beta\mu_j \mathcal{T}_t.$$
$$\mathcal{T}_{t+1} = \sum_j rr_{jt+1}\bar{\Theta}_{jt}$$

$\forall j \in \mathcal{J}_D$. This is the problem from the "timeless" perspective. An alternative recursive representation is to define $\boldsymbol{y}_t = \{\mathbb{E}_t\pi_{Nt+1}, n\tilde{f}a_t, \tilde{r}_{jt}\}$ and

$$\mathcal{V}(y_{t-1};\xi_{t-1}) = \max_{\{\boldsymbol{y}_t,rr_t,x_t,\mathcal{T}_t,\bar{\Theta}_{t-1}\}} -\frac{1}{2}\mathbb{E}_{t-1}\Bigg\{\left(A_{\mathcal{T}\mathcal{T}} + \left(\frac{1-\beta}{m}\right)(\frac{\partial U}{\partial C_T} - \frac{\partial U^*}{\partial C_T}\bar{\lambda})\gamma^*\right)(\mathcal{T}_t - \mathcal{T}_{cm,t})^2 \tag{237}$$
$$+ \lambda_x x_t^2 + \lambda_\pi \pi_{Nt}^2 + \frac{\partial U}{\partial C_T}\gamma_T\left(n\tilde{f}a_{t-1} - \beta n\tilde{f}a_t\right)^2 + \beta\mathcal{V}(y_t;\xi_t)\Bigg\} + \text{t.i.p.} + \mathcal{O}(\epsilon^3)$$

subject to

$$
\begin{aligned}
\pi_{Nt} &= \kappa x_t + \beta \mathbb{E}_t \pi_{Nt+1} \\
\mathbb{E}_{t-1} rr_{jt} &= 0, \\
\beta rr_{jt} - \beta rr_{jt}(0) &= \tilde{r}_{jt-1} - \beta(1-\delta_j)\tilde{r}_{jt} + k_{rr_j\pi}\pi_{Nt} + k_{rr_jx}x_t + \beta k_{rr_jb}\Delta \tilde{nfa}_t + \beta\mu_j \mathcal{T}_t. \\
\mathcal{T}_t &= \sum_j rr_{jt}\bar{\Theta}_{jt-1}
\end{aligned}
$$

In this case, the planner makes state-contingent decisions for t before uncertainty is realized; accordingly, it needs to respect a promise of expected rather than realized inflation. This representation is more convenient as it reduces the dimension of the state-space. Note that, by certainty equivalence, further innovations at $t \geq t_0 + 1$ are irrelevant for optimal decisions at t_0 and the portfolio chosen at $t_0 - 1$. Thus, w.l.o.g. I assume there is no uncertainty in periods $t \geq t_0 + 1$. Iterating forward on (237), note that one may write the objective as

$$
\mathcal{V}_{t_0} = \hat{\mathcal{V}}_{t_0} + \beta \mathbb{E}_{t_0-1}\mathcal{V}^{\dagger}_{t_0} + \text{t.i.p.} + \mathcal{O}(\epsilon^3)
$$

where

$$
\hat{\mathcal{V}}_t = -\frac{1}{2}\sum_{t\geq t_0}\beta^{t-t_0}\left\{\lambda_x\left(\mathbb{E}_{t_0-1}x_t\right)^2 + \lambda_\pi\left(\mathbb{E}_{t_0-1}\pi_{Nt}\right)^2 + \frac{\partial U}{\partial C_T}\gamma_T\left(\mathbb{E}_{t_0-1}\tilde{nfa}_{t-1} - \beta\mathbb{E}_{t_0-1}\tilde{nfa}_t\right)^2\right\} \tag{238}
$$

$$
\mathcal{V}^{\dagger}_t = -\frac{1}{2}\left(A_{\mathcal{T}\mathcal{T}} + \left(\frac{1-\beta}{m}\right)\left(\frac{\partial U}{\partial C_T} - \frac{\partial U^*}{\partial C_T}\bar{\lambda}\right)\gamma^*\right)\left(\mathcal{T}_{t_0} - \mathcal{T}_{cm,t_0}\right)^2 \tag{239}
$$
$$
-\frac{1}{2}\sum_{t\geq t_0}\beta^{t-t_0}\left\{\lambda_x\left(x^{\dagger}_t\right)^2 + \lambda_\pi\left(\pi^{\dagger}_{Nt}\right)^2 + \frac{\partial U}{\partial C_T}\gamma_T\left(\tilde{nfa}^{\dagger}_{t-1} - \beta\tilde{nfa}^{\dagger}_t\right)^2\right\},
$$

where $y^{\dagger}_t$ are time-t_0 innovations, i.e. $y^{\dagger}_t = y_t - \mathbb{E}_{t_0-1}y_t$. To solve the problem, I first study the optimal path for expectations $\{\mathbb{E}_{t_0-1}x_t, \mathbb{E}_{t_0-1}\pi_{Nt+1}, \mathbb{E}_{t_0-1}\tilde{nfa}_t, \mathbb{E}_{t_0-1}\tilde{r}_t\}_{t=t_0}^{\infty}$. Next, I study the optimal $t = t_0$ innovations $\{x^{\dagger}_{t_0}, \pi^{\dagger}_{Nt_0}, \tilde{nfa}^{\dagger}_{t_0}, \tilde{r}^{\dagger}_{t_0}\}$ that deliver a given set of promised returns rr_{t_0}. After this, the problem is similar to the static model and I solve for optimal promises of realized returns for $t = t_0$. Finally, I solve for the optimal portfolio at $t = t_0 - 1$.

B.4.7 Expectations

Taking $t_0 - 1$ expectations on the constraints (231)–(234),

$$
\kappa\mathbb{E}_{t_0-1}x_t + \beta\mathbb{E}_{t_0-1}\pi_{Nt+1} = \mathbb{E}_{t_0-1}\pi_{Nt} \tag{240}
$$
$$
\beta(1-\delta_j)\mathbb{E}_{t_0-1}\tilde{r}_{jt} - k_{rr_jx}\mathbb{E}_{t_0-1}x_t = \mathbb{E}_{t_0-1}\tilde{r}_{jt-1} + k_{rr_j\pi}\mathbb{E}_{t_0-1}\pi_{Nt} + \beta k_{rr_jb}\mathbb{E}_{t_0-1}\Delta\tilde{nfa}_t. \tag{241}
$$

The optimal path for expectations of inflation, output gaps, savings gaps, and prices $\{\mathbb{E}_{t_0-1}x_t$, $\mathbb{E}_{t_0-1}\pi_{Nt+1}$, $\mathbb{E}_{t_0-1}\tilde{nfa}_t$, $\mathbb{E}_{t_0-1}\tilde{r}_t\}_{t=t_0}^{\infty}$ maximizes (238) subject to (240) and (241) with $\tilde{r}_{jt_0-1}$, $\mathbb{E}_{t_0-1}\pi_{Nt_0}$, and $\tilde{nfa}_{t_0-1}$ given. Let $\beta^{t-t_0}\phi_t$ and $\beta^{t-t_0}\nu_{jt}$ denote the Lagrange multipliers on (240) and (241), respectively. The FOCs with respect to $\mathbb{E}_{t_0-1}x_t$, $\mathbb{E}_{t_0-1}\pi_{Nt+1}$, $\mathbb{E}_{t_0-1}\tilde{r}_{jt}$, and $\mathbb{E}_{t_0-1}\tilde{nfa}_t$ yield, respectively

$$\lambda_x \mathbb{E}_{t_0-1} x_t = \kappa \phi_t + \sum_{j \in \mathcal{J}_D} k_{rr_j x} \nu_{jt} \tag{242}$$

$$\lambda_\pi \mathbb{E}_{t_0-1} \pi_{Nt+1} + \phi_{t+1} - \sum_{j \in \mathcal{J}_D} k_{rr_j \pi} \nu_{jt+1} = \phi_t \tag{243}$$

$$\nu_{jt+1} = (1 - \delta_j) \nu_{jt} \tag{244}$$

$$\beta \frac{\partial U}{\partial C_T} \gamma_T \mathbb{E}_{t_0-1} n\tilde{f}a_{t+1} - \beta \sum_{j \in \mathcal{J}_D} k_{rr_j b} \nu_{jt+1} = (1+\beta) \frac{\partial U}{\partial C_T} \gamma_T \mathbb{E}_{t_0-1} n\tilde{f}a_t - \frac{\partial U}{\partial C_T} \gamma_T \mathbb{E}_{t_0-1} n\tilde{f}a_{t-1} - \sum_{j \in \mathcal{J}_D} k_{rr_j b} \nu_t \tag{245}$$

These four equations together with the constraints (240) and (241) form a system of equations in differences. There are $J+2$ roots outside the unit circle and a unit root.[58] Picking the initial condition $\{\phi_{t_0}, \{\nu_{jt_0}\}_{j \in \mathcal{J}_D}, n\tilde{f}a_{t_0}\}$ appropriately so that the system does not diverge, one obtains the solution at $t = t_0$,

$$\begin{aligned}
\mathbb{E}_{t_0-1} x_{t_0} &= \hat{k}_x[\{\tilde{r}_{jt_0-1}\}_j, \mathbb{E}_{t_0-1} \pi_{Nt_0}] \\
\mathbb{E}_{t_0-1} n\tilde{f}a_{t_0} &= \hat{k}_b[\{\tilde{r}_{jt_0-1}\}_j, \mathbb{E}_{t_0-1} \pi_{Nt_0}] + n\tilde{f}a_{t_0-1} \\
\mathbb{E}_{t_0-1} \pi_{Nt_0+1} &= \hat{k}_\pi[\{\tilde{r}_{jt_0-1}\}_j, \mathbb{E}_{t_0-1} \pi_{Nt_0}] \\
\mathbb{E}_{t_0-1} \tilde{r}_{jt_0} &= \hat{k}_r[\{\tilde{r}_{jt_0-1}\}_j, \mathbb{E}_{t_0-1} \pi_{Nt_0}].
\end{aligned}$$

Note that there is no interaction between $n\tilde{f}a_{t_0-1}$, the return terms $\{\tilde{r}_{jt_0-1}\}_j$, and $\mathbb{E}_{t_0-1}\pi_{Nt_0}$. This is easy to see from the system (240)−(245): if some path $\{\mathbb{E}_{t_0-1} x_t, \mathbb{E}_{t_0-1} \pi_{Nt+1}, \mathbb{E}_{t_0-1} n\tilde{f}a_t, \mathbb{E}_{t_0-1} \tilde{r}_t\}$ solves the problem for some $n\tilde{f}a_{t_0-1}$, then $\{\mathbb{E}_{t_0-1} x_t, \mathbb{E}_{t_0-1} \pi_{Nt+1}, \mathbb{E}_{t_0-1} n\tilde{f}a_t + \varepsilon, \mathbb{E}_{t_0-1} \tilde{r}_t\}$ solves the problem for $n\tilde{f}a_{t_0-1} = n\tilde{f}a_{t_0-1} + \varepsilon$.

Note that the solution is of the form

$$\hat{\mathcal{V}}_t = -\frac{1}{2} \left\{ A_{\pi\pi} \left(\mathbb{E}_{t_0-1} \pi_{Nt} \right)^2 + 2 \tilde{r}'_{t_0-1} A_{r\pi} \left(\mathbb{E}_{t_0-1} \pi_{Nt} \right) + \tilde{r}'_{t_0-1} A_{rr} \tilde{r}_{t_0-1} + A_{bb} n\tilde{f}a^2_{t_0-1} \right\}$$

where $A_{\pi\pi}$ and A_{bb} are positive scalars and A_{rr} is a positive definite matrix.

[58] More precisely, only long assets ($\delta_j < 1$) feature an exploding root. If an asset j is short ($\delta_j = 1$), then $\mathbb{E}_{t_0} \nu_{jt} = 0$ $\forall t > t_0$ and one can "drop" the asset-pricing constraint from the continuation problem, i.e. the problem for $t > t_0$. Intuitively, the expected asset price in future periods is irrelevant for welfare since agents are not exposed to it at $t = t_0$.

B.4.8 Innovations

The optimal time-t_0 innovations from $t \geq t_0 + 1$ onwards solve the same problem as the $t_0 - 1$ expectations from $t \geq t_0$ onwards. Replacing the solution into the objective (239),

$$\mathcal{V}_{t_0}^{\dagger} = -\frac{1}{2}\Bigg\{ \left(A_{\mathcal{T}\mathcal{T}} + \left(\frac{1-\beta}{m}\right)(\frac{\partial U}{\partial C_T} - \frac{\partial U^*}{\partial C_T}\bar{\lambda})\gamma^*\right)(\mathcal{T}_{t_0} - \mathcal{T}_{cm,t_0})^2 + \lambda_x \left(x_{t_0}^{\dagger}\right)^2 + \lambda_\pi \left(\pi_{Nt_0}^{\dagger}\right)^2 \tag{246}$$

$$+ \beta A_{\pi\pi}\left(\pi_{Nt_0+1}^{\dagger}\right)^2 + 2\beta\left(\tilde{r}_{t_0+1}^{\dagger}\right)' A_{r\pi}\pi_{Nt_0+1}^{\dagger} + \beta\left(\tilde{r}_{t_0}^{\dagger}\right)' A_{rr}\left(\tilde{r}_{t_0}^{\dagger}\right)$$

$$+ \left(\beta^2 \frac{\partial U}{\partial C_T}\gamma_T + \beta A_{bb}\right)\left(n\tilde{f}a_{t_0}^{\dagger}\right)^2 \Bigg\} + \text{t.i.p.} + \mathcal{O}(\epsilon^3)$$

Taking t_0- and $t_0 - 1$- expectations on the constraints (231)−(234) and subtracting,

$$\pi_{Nt_0}^{\dagger} = \kappa x_{t_0}^{\dagger} + \beta\pi_{Nt_0+1}^{\dagger} \tag{247}$$

$$\beta rr_{jt_0} - \beta rr_{jt_0}(0) - \beta\mu_j\mathcal{T}_{t_0} = -\beta(1-\delta_j)\tilde{r}_{jt_0}^{\dagger} + k_{rr_j\pi}\pi_{Nt_0}^{\dagger} + k_{rr_jx}x_{t_0}^{\dagger} + \beta k_{rr_jb}n\tilde{f}a_{t_0}^{\dagger}. \tag{248}$$

The innovation problem is to choose $\{x_{t_0}^{\dagger}, \pi_{Nt_0}^{\dagger}, \pi_{Nt_0+1}^{\dagger}, \{\tilde{r}_{jt_0}^{\dagger}\}_{j\in\mathcal{J}_D}, n\tilde{f}a_{t_0}^{\dagger}\}$ to maximize (246) subject to (247) and (248). Let $\phi_{t_0}^{\dagger}$ and $\nu_{jt_0}^{\dagger}$ denote the Lagrange multipliers of (247) and (248), respectively. The FOCs with respect to $x_{t_0}^{\dagger}$, $\pi_{t_0}^{\dagger}$, $\pi_{t_0+1}^{\dagger}$, $n\tilde{f}a_{t_0}^{\dagger}$, and $\tilde{r}_{j't_0}^{\dagger}$ yield, respectively,

$$-\lambda_x x_{t_0}^{\dagger} + \kappa\phi_{t_0}^{\dagger} + \sum_{j\in\mathcal{J}_D} k_{rr_jx}\nu_{jt_0}^{\dagger} = 0 \tag{249}$$

$$-\lambda_\pi \pi_{t_0}^{\dagger} - \phi_{t_0}^{\dagger} + \sum_{j\in\mathcal{J}_D} k_{rr_j\pi}\nu_{jt_0}^{\dagger} = 0 \tag{250}$$

$$-A_{\pi\pi}\pi_{t_0+1}^{\dagger} - \sum_{j\in\mathcal{J}_D} A_{\pi r_j}\tilde{r}_{jt_0}^{\dagger} + \phi_{t_0}^{\dagger} = 0 \tag{251}$$

$$-A_{bb}n\tilde{f}a_{t_0}^{\dagger} + \sum_{j\in\mathcal{J}_D} k_{rr_jb}\nu_{jt_0}^{\dagger} = 0 \tag{252}$$

$$-A_{\pi r_j}\pi_{t_0+1}^{\dagger} - \sum_{j'\in\mathcal{J}_D} A_{r_jr_{j'}}\tilde{r}_{j't_0}^{\dagger} - (1-\delta_j)\nu_{jt_0}^{\dagger} = 0. \tag{253}$$

The solution to this problem yields

$$\begin{aligned}
\pi_{Nt_0}^{\dagger} &= k_\pi^{\dagger}\left(\beta rr_{jt_0} - \beta rr_{jt_0}(0) - \beta\mu_j\mathcal{T}_{t_0}\right)_{j\in\mathcal{J}_D} \\
\pi_{Nt_0+1}^{\dagger} &= k_{F\pi}^{\dagger}\left(\beta rr_{jt_0} - \beta rr_{jt_0}(0) - \beta\mu_j\mathcal{T}_{t_0}\right)_{j\in\mathcal{J}_D} \\
x_{t_0}^{\dagger} &= k_x^{\dagger}\left(\beta rr_{jt_0} - \beta rr_{jt_0}(0) - \beta\mu_j\mathcal{T}_{t_0}\right)_{j\in\mathcal{J}_D} \\
n\tilde{f}a_{t_0}^{\dagger} &= k_b^{\dagger}\left(\beta rr_{jt_0} - \beta rr_{jt_0}(0) - \beta\mu_j\mathcal{T}_{t_0}\right)_{j\in\mathcal{J}_D} \\
\tilde{r}_{jt_0}^{\dagger} &= k_r^{\dagger}\left(\beta rr_{jt_0} - \beta rr_{jt_0}(0) - \beta\mu_j\mathcal{T}_{t_0}\right)_{j\in\mathcal{J}_D} \\
\phi_{t_0}^{\dagger} &= k_\phi^{\dagger}\left(\beta rr_{jt_0} - \beta rr_{jt_0}(0) - \beta\mu_j\mathcal{T}_{t_0}\right)_{j\in\mathcal{J}_D} \\
\nu_{t_0}^{\dagger} &= k_\nu^{\dagger}\left(\beta rr_{jt_0} - \beta rr_{jt_0}(0) - \beta\mu_j\mathcal{T}_{t_0}\right)_{j\in\mathcal{J}_D}
\end{aligned}$$

for some vector of constants $k_\pi^\dagger$, $k_{F\pi}^\dagger$, $k_x^\dagger$, $k_b^\dagger$, $k_r^\dagger$, $k_\phi^\dagger \in \mathbb{R}^{J_D}$ and a square positive definite matrix $k_\nu^\dagger \in \mathbb{R}^{J_D \times J_D}$. By the envelope theorem,

$$\begin{aligned}\mathcal{V}_{t_0}^\dagger &= -\frac{1}{2}\left(\beta rr_{jt_0} - \beta rr_{jt_0}(0) - \beta\mu_j \mathcal{T}_{t_0}\right)'_{j\in\mathcal{J}_D} k_\nu^\dagger \left(\beta rr_{jt_0} - \beta rr_{jt_0}(0) - \beta\mu_j \mathcal{T}_{t_0}\right)_{j\in\mathcal{J}_D} \\ &\quad - \frac{1}{2}\left(A_{\mathcal{T}\mathcal{T}} + \left(\frac{1-\beta}{m}\right)\left(\frac{\partial U}{\partial C_T} - \frac{\partial U^*}{\partial C_T}\bar{\lambda}\right)\gamma^*\right)\left(\mathcal{T}_{t_0} - \mathcal{T}_{cm,t_0}\right)^2 + \text{t.i.p.} + \mathcal{O}(\epsilon^3)\end{aligned} \tag{254}$$

B.4.9 Optimal returns

Next, I solve for the optimal combination of realized returns $\{rr_{jt_0}\}_{j\in\mathcal{J}_D}$ that delivers a given transfer $\mathcal{T}_{t_0}$ when agents hold portfolio $\bar{\Theta}_{t_0-1}$. Of course, this step is unnecessary if there is only one endogenous asset, i.e. if $J_D = 1$. When $J_D > 1$, I maximize (254) subject to $\sum_j rr_{jt_0}\bar{\Theta}_{jt_0-1} = \mathcal{T}_{t_0}$. This yields

$$\begin{aligned}\{rr_{jt_0} - rr_{jt_0}(0) - \mu_j\mathcal{T}_{t_0}\}_{j\in\mathcal{J}_D} = &-\left(k_\nu^\dagger\right)^{-1}\bar{\Theta}_{J_D t_0-1}\left(\frac{1 - \bar{\Theta}'_{\mathcal{J}_D t_0-1}\mu_{\mathcal{J}_D}}{\bar{\Theta}'_{\mathcal{J}_D t_0-1}\left(k_\nu^\dagger\right)^{-1}\bar{\Theta}_{\mathcal{J}_D t_0-1}}\right) \\ &\times\left(\mathcal{T}_{t_0} - \frac{\bar{\Theta}'_{t_0-1}}{1 - \bar{\Theta}'_{\mathcal{J}_D t_0-1}\mu_{\mathcal{J}_D}} rr_{t_0}(0)\right)\end{aligned}$$

where I use the notation that $y_\mathcal{J} = \{y_j\}_{j\in\mathcal{J}}$ for a generic vector y and set $\mathcal{J}$ of natural numbers. Replacing into (254) I obtain

$$\mathbb{E}_{t_0-1}\mathcal{V}_{t_0}^\dagger = -\frac{1}{2}\frac{k_0}{\tilde{\Theta}'_{\mathcal{J}_D t_0-1}\chi\tilde{\Theta}_{\mathcal{J}_D t_0-1}}\mathbb{E}_{t_0-1}\left\{\left(\tilde{\Theta}'_{\mathcal{J}_D t_0-1}\chi\tilde{\Theta}_{\mathcal{J}_D t_0-1}\right)\left(\mathcal{T}_{t_0} - \mathcal{T}_{cm,t_0}\right)^2 + \left(\mathcal{T}_{t_0} - \tilde{\Theta}'_{t_0-1}rr_{t_0}(0)\right)^2\right\} \tag{255}$$

where

$$\begin{aligned}k_0 &= \left(A_{\mathcal{T}\mathcal{T}} + \left(\frac{1-\beta}{m}\right)\left(\frac{\partial U}{\partial C_T} - \frac{\partial U^*}{\partial C_T}\bar{\lambda}\right)\gamma^*\right) \\ \chi &= \beta^{-2}\left(A_{\mathcal{T}\mathcal{T}} + \left(\frac{1-\beta}{m}\right)\left(\frac{\partial U}{\partial C_T} - \frac{\partial U^*}{\partial C_T}\bar{\lambda}\right)\gamma^*\right)\left(k_\nu^\dagger\right)^{-1} \\ \tilde{\Theta}_{jt_0-1} &= \left(\frac{1}{1 - \bar{\Theta}'_{\mathcal{J}_D t_0-1}\mu_{\mathcal{J}_D}}\right)\bar{\Theta}_{jt_0-1}\end{aligned} \tag{256}$$

Intuitively, χ controls how expensive it is for the planner to deviate from the demand-management policy after choosing the optimal combination of output gaps, inflation, and savings distortions. Indeed, when $J_D = 1$, the problem is isomorphic to the one in the static model.

B.4.10 Optimal transfers

The optimal transfers $\mathcal{T}_{t_0}$ maximize (255). This yields,

$$\mathcal{T}_{t_0} = \frac{\tilde{\Theta}'_{\mathcal{J}_D t_0-1}\chi\tilde{\Theta}_{\mathcal{J}_D t_0-1}}{1 + \tilde{\Theta}'_{\mathcal{J}_D t_0-1}\chi\tilde{\Theta}_{\mathcal{J}_D t_0-1}}\mathcal{T}_{cm,t_0} + \frac{1}{1 + \tilde{\Theta}'_{\mathcal{J}_D t_0-1}\chi\tilde{\Theta}_{\mathcal{J}_D t_0-1}}\tilde{\Theta}' rr_{t_0}(0) \tag{257}$$

Replacing back in (255), and taking time $t = t_0 - 1$ expectations:

$$\mathbb{E}_{t_0-1}\mathcal{V}^{\dagger}_{t_0} = -\frac{1}{2}\frac{k_0}{1+\tilde{\Theta}'_{\mathcal{J}_D t_0-1}\chi\tilde{\Theta}_{\mathcal{J}_D t_0-1}}\Bigg\{\sigma^2_{\mathcal{T}_{cm,t_0}} + \tilde{\Theta}'_{t_0-1}\text{Var}(rr_{t_0}(0))\tilde{\Theta}_{t_0-1} - 2\tilde{\Theta}'_{t_0-1}\text{Cov}(rr_{t_0}(0), \mathcal{T}_{cm,t_0})\Bigg\}. \tag{258}$$

B.4.11 Proposition 19: Optimal portfolios

Here, I prove an analogue of proposition 10 and lemma 5.

Proposition 19. *Given some position in endogenous assets $\bar{\Theta}_{\mathcal{J}_D,t_0-1}$ the optimal portfolio on exogenous assets solves*

$$\bar{\Theta}_{\mathcal{J}_X t_0-1} = \left(1-\bar{\Theta}'_{\mathcal{J}_D t_0-1}\mu_{\mathcal{J}_D}\right) Var(rr_{\mathcal{J}_X,t_0}(0))^{-1}\times \left\{Cov(rr_{\mathcal{J}_X t_0}(0), \mathcal{T}_{cm,t_0}) - Cov(rr_{\mathcal{J}_X t_0}(0), rr_{\mathcal{J}_D t_0}(0))\frac{\bar{\Theta}_{\mathcal{J}_D,t_0-1}}{1-\bar{\Theta}'_{\mathcal{J}_D t_0-1}\mu_{\mathcal{J}_D}}\right\}.$$

Taking the first-order condition of (258) with respect to $\bar{\Theta}_{\mathcal{J}_X,t_0-1}$ and rearranging yields the desired result.

Lemma 11. *The optimal portfolio on endogenous assets solves*

$$max - \frac{1}{2}k_0\left(\frac{1}{1+\tilde{\Theta}'_{\mathcal{J}_D t_0-1}\chi\tilde{\Theta}_{\mathcal{J}_D t_0-1}}\right)\Bigg\{\sigma^2_{\tilde{\mathcal{T}}_{cmt_0}} + \tilde{\Theta}'_{\mathcal{J}_D t_0-1}Var(\tilde{rr}_{\mathcal{J}_D t_0}(0))\tilde{\Theta}_{\mathcal{J}_D t_0-1} - 2\tilde{\Theta}'_{\mathcal{J}_D t_0-1}Cov(\tilde{rr}_{\mathcal{J}_D,t_0}(0), \mathcal{T}_{cm,t_0})\Bigg\} \tag{259}$$

where

$$\tilde{\mathcal{T}}_{cm,t_0} = \mathcal{T}_{cm,t_0} - Cov(\tilde{rr}_{\mathcal{J}_X t_0}(0), \mathcal{T}_{cm,t_0})' Var(\tilde{rr}_{\mathcal{J}_X t_0}(0))^{-1} rr_{\mathcal{J}_X t_0}(0),$$
$$\tilde{rr}_{\mathcal{J}_D t_0}(0) = rr_{\mathcal{J}_D t_0}(0) - Cov(\tilde{rr}_{\mathcal{J}_X t_0}(0), \tilde{rr}_{\mathcal{J}_D t_0})' Var(\tilde{rr}_{\mathcal{J}_X t_0}(0))^{-1} rr_{\mathcal{J}_X t_0}(0).$$

Note that, if $\#(\mathcal{J}_D) = 1$, this simplifies to the expression in lemma 5.

Replacing the result of proposition 19 into (258) yields the desired result.

B.4.12 Proposition 20: Robustness when there is a single endogenous asset

Here, I prove an analogue of proposition 11.

Proposition 20. *If there is a single endogenous asset, i.e. $\#(\mathcal{J}_D) = 1$, then proposition 11 carries over to the dynamic model unaltered with $f(\bar{\Theta}_{t_0-1}) = \tilde{\Theta}_{\mathcal{J}_D t_0-1}$ and returns $rr_{ft_0} = f(\bar{\Theta}_{t_0-1})^{-1}\tilde{\mathcal{T}}_{cm,t_0}$.*

This result follows immediately from the fact that the optimal transfer (257) and the objective function (259) have the same form as in the static model. By contrast, when $\#(\mathcal{J}_D) > 1$, (259) is more complicated since $\tilde{\Theta}$ and χ are multidimensional and, hence, there is not a closed form solution. As a result, one cannot define a single-dimensional measure of exposure to monetary policy $f(\bar{\Theta})$ or its returns rr_f.

B.4.13 Solving when there is more than one endogenous asset

When $J_D > 1$ there is no closed form solution for the portfolio problem. However, there are a finite number of solutions. To see this, define $f(\bar{\Theta}_{t_0-1}) = \tilde{\Theta}'_{\mathcal{J}_D t_0-1}\chi\tilde{\Theta}_{\mathcal{J}_D t_0-1}$ and solve the problem conditional on an exposure to monetary policy $f(\bar{\Theta}_{t_0-1})$. This yields

$$\begin{aligned}\mathrm{Var}(\tilde{rr}_{\mathcal{J}_D t_0})\tilde{\Theta}_{\mathcal{J}_D t_0-1} - \mathrm{Cov}(\tilde{rr}_{\mathcal{J}_D t_0}, \tilde{\mathcal{T}}_{cm,t_0}) - \tilde{\lambda}\chi\tilde{\Theta}_{\mathcal{J}_D t_0-1} &= 0\\ \tilde{\Theta}'_{\mathcal{J}_D t_0-1}\chi\tilde{\Theta}_{\mathcal{J}_D t_0-1} &= f(\bar{\Theta}_{t_0-1}).\end{aligned}$$

Solve for $\tilde{\Theta}_{\mathcal{J}_D t_0-1}$,

$$\tilde{\Theta}_{\mathcal{J}_D t_0-1} = (\mathrm{Var}(\tilde{rr}_{\mathcal{J}_D t_0}) - \tilde{\lambda}\chi)^{-1}\mathrm{Cov}(\tilde{rr}_{\mathcal{J}_D t_0}, \tilde{\mathcal{T}}_{cm,t_0})$$

and replace to obtain an equation in $\tilde{\lambda}$,

$$\mathrm{Cov}(\tilde{rr}_{\mathcal{J}_D t_0}, \tilde{\mathcal{T}}_{cm,t_0})'(\mathrm{Var}(\tilde{rr}_{\mathcal{J}_D t_0}) - \tilde{\lambda}\chi)^{-1}\chi(\mathrm{Var}(\tilde{rr}_{\mathcal{J}_D t_0}) - \tilde{\lambda}\chi)^{-1}\mathrm{Cov}(\tilde{rr}_{\mathcal{J}_D t_0}, \tilde{\mathcal{T}}_{cm,t_0}) = f(\bar{\Theta}_{t_0-1}).$$

Note this can be written as

$$\frac{\mathcal{P}_1(\tilde{\lambda})}{(\mathcal{P}_2(\tilde{\lambda}))^2} = f(\bar{\Theta}_{t_0-1})$$

where $\mathcal{P}_1(\tilde{\lambda})$ is a polynomial of degree $(J_D-1)^2$ and $\mathcal{P}_2(\tilde{\lambda})$ is a polynomial degree J_D. Thus, there are at most J_D^2 solutions which need to be checked. Using this and then maximizing over $f(\bar{\Theta})$ one can compute the optimal portfolios. Unfortunately, $\tilde{\Theta}$ is nonlinear in $f(\bar{\Theta})$ if $J_D > 1$, so there is no analogue of proposition 11 in this case.

B.4.14 Proposition 21: Optimal portfolio taxes

Proposition 21. *In an interior optimum, the optimal tax on asset j relative to the risk-free asset is given by*

$$\tau_{jt_0-1} - \tau_{0t_0-1} = \left(\frac{1-\beta}{m}\right)\left(1 - \frac{\frac{\partial U^*}{\partial C_T}}{\frac{\partial U}{\partial C_T}}\bar{\lambda}\right)\gamma^*\,Cov(\mathcal{T}_{t_0}, rr_{jt_0}) + \mathcal{O}(\epsilon^3).$$

To prove the result, it is helpful to take a step back and consider the first-order conditions with respect to the realized returns $\{rr_{t_0}\}$, the transfers $\{\mathcal{T}_{t_0}\}$ and the portfolio $\bar{\Theta}_{t_0-1}$ before replacing the optimal innovations. That is, maximizing the expected value of (246) subject to (247), (248) and the definition of the transfer,

$$\mathcal{T}_{t_0} = \bar{\Theta}' rr_{t_0}.$$

Let λ_{t_0} denote the Lagrange multiplier on this constraint. The FOCs with respect to rr_{j,t_0}, $\mathcal{T}_{t_0}$ and $\bar{\Theta}_{t_0-1}$ yield

$$-\beta\nu^{\dagger}_{jt_0} + \bar{\Theta}_{jt_0-1}\lambda_{t_0} = 0 \tag{260}$$

$$-\left(A_{\mathcal{T}\mathcal{T}} + \left(\frac{1-\beta}{m}\right)\left(\frac{\partial U}{\partial C_T} - \frac{\partial U^*}{\partial C_T}\bar{\lambda}\right)\gamma^*\right)(\mathcal{T}_{t_0} - \mathcal{T}_{cm,t_0}) - \lambda_{t_0} + \beta\sum_j \mu_j\nu^{\dagger}_{t_0 j} = 0 \tag{261}$$

$$\mathbb{E}_{t_0-1} rr_{jt_0}\lambda_{t_0} = 0 \tag{262}$$

Next, note that the FOC (249) - (253) imply that one can write the optimal innovations

$\{x^\dagger_{t_0}, \pi^\dagger_{Nt_0}, \pi^\dagger_{Nt_0+1}, \{\tilde{r}^\dagger_{jt_0}\}_{j\in\mathcal{J}_D}, n\tilde{f}a^\dagger_{t_0}\}$ as a function of $\{\nu^\dagger_{jt_0}\}_j$, e.g.

$$x^\dagger_{t_0} = \sum_j \tilde{k}_{xj}\nu^\dagger_{jt_0}$$

Using (260), one obtains that the optimal innovations are proportional to λ_{t_0}, e.g.

$$x^\dagger_{t_0} = \beta^{-1}\left(\sum_j \tilde{k}_{xj}\bar{\Theta}^{-1}_{jt_0-1}\right)\lambda_{t_0}$$

Intuitively, this equation reflects the fact that the planner introduces distortions into the economy to improve insurance. Therefore, the portfolio optimality condition (262) implies that the optimal innovations are uncorrelated with the returns of the asset, e.g.

$$\mathbb{E}_{t_0-1}x^\dagger_{t_0}rr_{t_0} = 0.$$

Similarly, I obtain that the remaining innovations $\{\pi^\dagger_{Nt_0}, \pi^\dagger_{Nt_0+1}, \{\tilde{r}^\dagger_{jt_0}\}_{j\in\mathcal{J}_D}, n\tilde{f}a^\dagger_{t_0}\}$ are uncorrelated with the returns of the assets. For future reference, note that replacing (261) and the definition of $\mathcal{T}_{cm}$ (199) into the portfolio optimality condition (262) yields

$$\left(A_{\mathcal{T}\mathcal{T}} + \left(\frac{1-\beta}{m}\right)\left(\frac{\partial U}{\partial C_T} - \frac{\partial U^*}{\partial C_T}\bar{\lambda}\right)\gamma^*\right)\mathbb{E}_{t_0-1}\mathcal{T}_{t_0}rr_{jt_0} - \mathcal{A}_{\mathcal{T}\xi}\mathbb{E}_{t_0-1}\epsilon_{t_0}rr_{jt_0} = 0. \tag{263}$$

Next, I combine a second-order approximation of home and foreign private marginal utility to obtain:

$$\underbrace{\begin{aligned}&-A_{\mathcal{T}\mathcal{T}}\mathbb{E}_{t_0-1}\mathcal{T}_{t_0}rr_{jt_0} + A_{\mathcal{T}\xi}\mathbb{E}_{t_0-1}\epsilon_{t_0}rr_{jt_0}\\ &+\beta\left(\frac{\partial U}{\partial C_T}\right)\gamma_T\mathbb{E}_{t_0-1}n\tilde{f}a^\dagger_{t_0}rr_{jt_0} + \left(\frac{\partial F}{\partial L}\right)^{-1}A_{cl}x^\dagger_{t_0}rr_{jt_0}\end{aligned}}_{=0} = \left(\frac{\partial U}{\partial C_T}\right)(\tau_{jt_0-1} - \tau_{0t_0-1}) + \mathcal{O}(\epsilon^3) \tag{264}$$

Replacing (263) into (264) and rearranging yields the desired result. The intuition is similar to the static model: the distortions (now both savings & output gaps) introduce wedges between private and social marginal utility. However, these wedges are related to the value of insurance and, therefore, they are uncorrelated with the return of every available asset under the optimal policy.

B.4.15 The role of time-varying capital controls

As discussed in section 5, the planner does not want to distort portfolio decisions in the limit, but they do want to manipulate savings decisions, i.e. put time-varying taxes that are the same for all assets. Indeed, one may back out the optimal taxes from a first-order approximation of the Euler equation (187):

$$\underbrace{-\gamma_T\left(\Delta n\tilde{f}a_t - \beta\mathbb{E}_t\Delta n\tilde{f}a_{t+1}\right)}_{\text{pecuniary externality}} + \underbrace{\bar{C}_N\left(\frac{\partial U}{\partial C_T}\right)^{-1}\left(\frac{\partial F}{\partial L}\right)^{-1}A_{cl}\mathbb{E}_t\Delta x_{t+1}}_{\text{aggregate-demand externality}} = \tau_{0t}.$$

Interestingly, A_{cl} may take different signs for standard utility functions (see equation 119). If U is GHH with a CES tradable-nontradable aggregator and CRRA with respect to the composite, then

$$\left(\frac{\partial U}{\partial C_T}\right)^{-1} A_{cl} = \frac{1}{\rho} > 0,$$

where ρ is the elasticity of substitution between tradables and nontradables. Thus, agents always overvalue tradable goods in booms. Instead, if U is separable in labor,

$$\left(\frac{\partial U}{\partial C_T}\right)^{-1} A_{cl} = \frac{\rho\gamma - 1}{\rho},$$

where γ is the CRRA risk-aversion parameter. Thus, depending on whether tradables and nontradables are Edgeworth complements ($\rho\gamma < 1$) or substitutes ($\gamma\rho > 1$), agents will under- or over-value tradable goods in recessions, respectively, yielding potentially opposite predictions on savings taxes.

The theoretical results described in this appendix also extend to a setting where the planner can only put time-invariant taxes on financial assets, i.e. the planner cannot manipulate private savings.[59] In such a case, the only difference is that one should add

$$-\gamma_T \left(\Delta n\tilde{f}a_t - \beta \mathbb{E}_t \Delta n\tilde{f}a_{t+1}\right) + \bar{C}_N \left(\frac{\partial U}{\partial C_T}\right)^{-1} \left(\frac{\partial F}{\partial L}\right)^{-1} A_{cl} \mathbb{E}_t \Delta x_{t+1} = 0$$

as a constraint in the problems analyzed in sections B.4.7 and B.4.8. Naturally, since there is an additional constraint in the ability of the planner to provide insurance, the cost of deviating from demand-management, controlled by $\tilde{\chi}$ at the end of that section, will be higher. The remainder of the analysis is isomorphic. Importantly, note that the planner would still use the same time-invariant asset-specific taxes $\tau_j - \tau_0$ to control steady-state portfolios. One may think of this case as one where controls are "sticky", i.e. the planner cannot move taxes over the business cycle, but it can put time-invariant controls. In appendix C.4, I compare the solution with and without these taxes in the calibrated model.

B.5 Examples of non-zero approximate taxes

In this section, I study two extensions of the general model of section 4 that illustrate common reasons why the approximate no-tax result may not hold. To emphasize the role of the multi-dimensional aggregate-demand externality, I focus on the small-open-economy case $m \to \infty$. The two extensions share one critical feature: the aggregate-demand externality is multidimensional but the planner has no additional tools. Despite this, in both cases the result on taxes is subtle and crucially relies on non-separability between tradable and non-tradable goods. When they are separable, the tax is still approximately zero.[60]

The first extension studies a model with mark-up shocks. Mark-up shocks create variation in output gaps that is unrelated to the value of insurance. The optimal policy prescribes booms when mark ups are low and recessions when mark ups are high to alleviate the cost of inefficient price dispersion. As a result, even under complete markets, asset returns may be correlated with output gaps if they are correlated with mark-up shocks. I show that a standard model with GHH

[59] In a previous version of this paper, I characterized more explicitly the solution to this problem. Since it is very similar, the algebra is omitted from this version for brevity.

[60] Separability here should be interpreted broadly. For ease of exposition I only introduce non-separability in preferences. However, a model with separable preferences and non-separabilities in production would be similar to a model with non-separable preferences.

preferences implies that agents overvalue tradable goods in booms. Thus, assets that pay in states where mark ups are high are undervalued.

The second extension studies an economy with multiple non-tradable sectors, each with their own nominal rigidity. In this case, the argument for taxes is more subtle: even with GHH utility taxes are zero if prices are fully rigid. I present two simple examples that give rise to a non-zero tax. The first example features a "mixed" utility function where one nontradable good is separable from tradable consumption and the other is not. The second example features GHH utility but heterogeneous degrees of price stickiness across sectors.

B.5.1 Mark-up shocks

In this section, I extend the model of section 4 to allow for shocks to the elasticity of substitution across varieties. The only equilibrium relationship that changes is the optimality condition of flexible-price firms (113), which now becomes

$$P_{Ns}(i) = \mathcal{M}(\xi_s)(1-\tau_L)\frac{1}{\frac{\partial F}{\partial L}(s)}W_s \text{ for } i \in \{\text{flex}\}, \tag{265}$$

where $\mathcal{M}(\xi_s) = \frac{\eta(\xi_s)}{\eta(\xi_s)-1}$ is the desired mark up. I assume the labor subsidy is such that the economy is efficient at the steady state, i.e. $\tau_l = (\bar{\eta}-1)/\bar{\eta}$.

Following the same steps as before, one may write the approximate planner's problem as

$$\max_{\{\mathcal{T}_s,\{rr_{js}\}_j,x_s\}_s} -\sum_s \frac{1}{2}\pi_s\left\{A_{\mathcal{T}\mathcal{T}}\left(\mathcal{T}_s-\mathcal{T}_{cm,s}\right)^2 + \left(\kappa\lambda_\pi+\lambda_x\right)\left(x_s-\tilde{x}_s\right)^2\right\} + \text{t.i.p.} + \mathcal{O}(\epsilon^3)$$

subject to

$$\mathcal{T}_s = \sum_j \bar{\Theta}_j rr_{js}$$
$$rr_{js} = k_{rr_j\mathcal{T}}\mathcal{T}_s + k_{rr_jx}x_s + \sum_k k_{rr_jk}\xi_{ks}.$$

The key difference is that, because of mark-up shocks, the planner does not seek to stabilize output gaps. Rather, when mark ups are high, the planner avoids high inflation by creating a recession:

$$\tilde{x}_s = -\left(\frac{\lambda_\pi\kappa\frac{\phi}{1-\phi}}{\lambda_\pi\kappa^2+\lambda_x}\right)D_\xi \ln\mathcal{M}\cdot\xi_s + \mathcal{O}(\epsilon^2). \tag{266}$$

This is a source of "exogenous" variation in output gaps; i.e. output gaps that do not reflect a desire to provide insurance. As a result, output gaps are generically correlated with asset returns to first order. Indeed, the FOCs of this problem imply

$$\sum_s \pi_s(x_s-\tilde{x}_s)rr_{js} = \mathcal{O}(\epsilon^3). \tag{267}$$

Furthermore, following the same steps as before, portfolio optimality also implies

$$-A_{\mathcal{T}\mathcal{T}}\sum_s \pi_s\mathcal{T}_s rr_{js} + \sum_s \pi_s rr_{js}A_{\mathcal{T}\xi}\cdot\xi_s = \mathcal{O}(\epsilon^3).$$

On the other hand, the expansion of home and foreign no-arbitrage equations is still given by

$$-\sum_s \pi_s A_{\mathcal{T}\mathcal{T}} \mathcal{T}_s rr_{js} + \sum_s \pi_s \left(\frac{\partial F}{\partial L}\right)^{-1} \bar{C}_N A_{cl} rr_{js} x_s + \sum_s \pi_s rr_{js} A_{\mathcal{T}\xi} \cdot \xi_s = \frac{\partial U}{\partial C_T} \tau_j + \mathcal{O}(\epsilon^3).$$

Thus, the tax of asset j is given by:

$$\tau_j = \left(\frac{\partial U}{\partial C_T}\right)^{-1} \sum_s \pi_s \left(\frac{\partial F}{\partial L}\right)^{-1} \bar{C}_N A_{cl} rr_{js} \tilde{x}_s + \mathcal{O}(\epsilon^3). \tag{268}$$

Note that, in any model where utility is separable between tradables and non-tradables, agents do not over- or undervalue goods in booms and recessions to first order under the optimal monetary policy ($A_{cl} = 0$).[61] Thus, the tax would be zero. With GHH, assets that pay in states where mark ups are high, which induce recessions $\tilde{x} < 0$, are undervalued by the private sector and need to be subsidized. Proposition 22 collects these results.

Proposition 22. *Consider an extension of the model of section 4 to allow for mark-up shocks (described above) and $m \to \infty$. In such a model, the optimal policy implies (267): the return of any tradable asset j must be uncorrelated with output gap deviations from the target output gap $\tilde{x}$, given by (266). The optimal tax τ_j is given by (268) and A_{cl} is given by (119). In particular, a model with separable utility implies $A_{cl} = 0$ and, hence, zero approximate taxes. A model with GHH preferences*

$$U(C_{Ts}, C_{Ns}, L_s) = \frac{1}{1-\gamma}\left(\left(\alpha^{\frac{1}{\rho}} C_{Ts}^{\frac{\rho-1}{\rho}} + (1-\alpha)^{\frac{1}{\rho}} C_{Ns}^{\frac{\rho-1}{\rho}}\right)^{\frac{\rho}{\rho-1}} - \frac{\alpha}{1+\varphi} L_s^{1+\varphi}\right)^{\frac{1}{1-\gamma}}$$

implies $A_{cl} = \frac{(1-\alpha)(1+\varphi)^\gamma}{\rho(\alpha+\varphi)^\gamma}$. In that case, assets with returns that are positively correlated with mark ups are undervalued ($\tau_j < 0$) and vice versa.

B.5.2 Multiple sources of nominal rigidities

In this section, I extend the general model of section 4 to allow for $M \geq 1$ nontradable goods. Formally, utility is now given by

$$\sum_s \pi_s U(C_{Ts}, \{C_{ms}\}_m, \{L_{ms}\}_m; \xi_s),$$

where U is assumed to be locally analytic and concave. Each nontradable good is a composite of a CES continuum of varieties

$$C_{ms} = (\int_0^1 C_{ms}(i)^{\frac{\eta_m - 1}{\eta_m}} di)^{\frac{\eta_m}{\eta_m - 1}}.$$

For each variety, there is a firm that produces it with labor,

$$C_{ms}(i) = F_m\left(L_{ms}(i); \xi_s\right).$$

Note that all varieties within a sector have the same technology. Thus, in the first best all firms in the same sector produce equal amounts.

[61] Note that the fact that monetary policy is optimal is important for this result. Sub-optimal monetary policy, e.g. a peg, could introduce another link between private consumption and the output gap. The key observation is that manipulating tradable consumption does not alleviate the trade off between price dispersion and output gaps.

The only role of this special structure is to introduce nominal rigidities into the environment. More precisely, I assume that in each state of the world a random share ϕ_m of the firms have a fixed home-currency price of $\bar{P}_m$ while the remaining share $1-\phi_m$ can reset their price. I assume there is a constant sector-specific labor subsidy $\tau_m^L = 1 - \frac{\eta_m - 1}{\eta_m}$ to correct the monopolistic distortion. Steady-state prices $\bar{P}_m$ are chosen such that the steady state is efficient, i.e.

$$\frac{\partial F_m}{\partial L_m}\frac{\partial U}{\partial C_m} = -\frac{\partial U}{\partial L_m}.$$

Second-order approximation to utility A second-order approximation of the utility flow yields

$$\begin{aligned} U_s = &-\frac{1}{2}p_s'\Lambda_\pi p_s + \frac{\partial U}{\partial C_T}\bar{C}_T\left(c_{Ts}+\frac{1}{2}c_{Ts}^2\right) + \frac{1}{2}\frac{\partial^2 U}{\partial C_T^2}\bar{C}_T^2 c_{Ts}^2 + (\bar{L}l)'A_{lc}\bar{C}_T c_{Ts} \\ &+ \bar{C}_T c_{Ts} A_{c\xi}\xi_s - \frac{1}{2}(\bar{L}l)'A_{ll}(\bar{L}l) + (\bar{L}l)'A_{l\xi}\cdot\xi_s + \text{t.i.p.} + \mathcal{O}(\epsilon^3) \end{aligned} \tag{269}$$

where $A_{ll}, A_{lz} \in \mathbb{R}^{M\times M}$, $A_{lc}\in\mathbb{R}^{M\times 1}$, $A_{zc}\in\mathbb{R}^{M\times 1}$, $\Lambda_\pi \in \mathbb{R}^{M\times M}$ and

$$\begin{aligned} A_{lc}(m) &= \frac{\partial^2 U}{\partial C_m \partial C_T}\frac{\partial F_m}{\partial L_m} + \frac{\partial^2 U}{\partial L_m \partial C_T} \\ A_{c\xi} &= D^2_{C_T\xi}U + \sum_m \frac{\partial^2 U}{\partial C_m\partial C_T}D_\xi F_m \\ A_{ll}(m,m') &= -\left(\mathbf{1}_{m=m'}\frac{\partial U}{\partial C_m}\frac{\partial^2 F_m}{\partial L_m^2} + \frac{\partial^2 U}{\partial C_m\partial C_{m'}}\frac{\partial F_m}{\partial L_m}\frac{\partial F_{m'}}{\partial L_{m'}} + \frac{\partial^2 U}{\partial C_m \partial L_{m'}}\frac{\partial F_m}{\partial L_m} + \frac{\partial^2 U}{\partial L_m\partial C_{m'}}\frac{\partial F_{m'}}{\partial L_{m'}} + \frac{\partial^2 U}{\partial L_m \partial L_{m'}}\right) \\ A_{l\xi}(m,:) &= \mathbf{1}_{m=m'}\frac{\partial U}{\partial C_m}D^2_{L_m\xi}F_m + \frac{\partial^2 U}{\partial C_m\partial C_{m'}}\frac{\partial F_m}{\partial L_m}D_\xi F_{m'} + \frac{\partial^2 U}{\partial L_m \partial C_{m'}}D_\xi F_{m'} + \frac{\partial F_m}{\partial L_m}D^2_{C_m\xi}U + D^2_{L_m\xi}U \\ \Lambda_\pi(m,m') &= \mathbf{1}_{m=m'}\frac{\partial U}{\partial C_m}\bar{C}_m\eta_m\left(\frac{\phi_m}{1-\phi_m}\right)\left(1-\frac{\bar{C}_m\frac{\partial^2 F_m}{\partial L_m^2}\eta_m}{(\frac{\partial F_m}{\partial L_m})^2}\right), \end{aligned}$$

and I abuse notation by writing $(\bar{Y}y)$ to denote $\{\bar{Y}_m y_m\}_{m=1}^M$ for an arbitrary variable $y\in\mathbb{R}^M$. If prices were flexible,

$$(\bar{L}l)^{\text{flex}} = A_{ll}^{-1}A_{l\xi}\cdot\xi_s + A_{ll}^{-1}A_{lc}\bar{C}_T c_{Ts} + \mathcal{O}(\epsilon^2).$$

Let $x_{ms} = \bar{C}_m^{-1}\frac{\partial F_m}{\partial L_m}\left(\bar{L}_m l_{ms} - \bar{L}_m l_{ms}^{\text{flex}}\right)$ denote the output gap in good m, and define a diagonal matrix $\mathcal{D}_1 \in \mathbb{R}^{M\times M}$ with

$$\mathcal{D}_1(m,m') = \mathbf{1}_{m=m'}\left(\bar{C}_m^{-1}(\frac{\partial F_m}{\partial L_m})\right).$$

Thus, one can rewrite the above as

$$\begin{aligned} U_s = &\frac{\partial U}{\partial C_T}\bar{C}_T\left(c_{Ts}+\frac{1}{2}c_{Ts}^2\right) - \frac{1}{2}A_{\mathcal{T}\mathcal{T}}\bar{C}_T^2c_{Ts}^2 + \bar{C}_T c_{Ts}\left(A_{c\xi} + A_{lc}'A_{ll}^{-1}A_{l\xi}\right)\xi_s \\ &-\frac{1}{2}p_s'\Lambda_\pi p_s - \frac{1}{2}x_s'\Lambda_x x_s + \text{t.i.p.} + \mathcal{O}(\epsilon^3) \end{aligned}$$

where

$$A_{\mathcal{T}\mathcal{T}} = -\left(\frac{\partial^2 U}{\partial C_T^2} + A'_{lc} A_{ll}^{-1} A_{lc}\right)$$
$$\Lambda_x = \mathcal{D}_1^{-1} A_{ll} \mathcal{D}_1^{-1}.$$

Next, note that consumer optimization implies

$$-\frac{\partial U}{\partial L_m} = \frac{W_{ms}}{P_{ms}} \frac{\partial U}{\partial C_m},$$

which to first order is equal to

$$-A_{lc}(m)\bar{C}_T c_{Ts} + \frac{\partial U}{\partial C_m} \frac{\partial^2 F_m}{\partial L_m^2} \bar{L}_m l_{ms} + A_{ll}(m,:)(\bar{L} l_s) = \tag{270}$$
$$(-\frac{\partial U}{\partial L_m})(w_{ms} - p_{ms}) - \frac{\partial U}{\partial C_m} D^2_{L_m \xi} F_m \cdot \xi_s + A_{l\xi}(m,:) \cdot \xi_s + \mathcal{O}(\epsilon^2)$$

A first-order approximation of flexible-firm's optimality condition yields,

$$p_{ms}(i) + \left(\frac{\partial F_m}{\partial L_m}\right)^{-1} \frac{\partial^2 F_m}{\partial L_m^2} \bar{L}_m l_{ms}(i) + \left(\frac{\partial F_m}{\partial L_m}\right)^{-1} D^2_{L\xi} F_m \cdot \xi_s = w_{ms} + \mathcal{O}(\epsilon^2).$$

Using this equation to substitute out w_{ms} in (270),

$$-A_{lc}(m)\bar{C}_T c_{Ts} + \sum_{m'} A_{ll}(m,m')\bar{L}_{m'} l_{ms}(i) = (-U_{L_m}) p_{ms}(i) + A_{l\xi}(m,:) \cdot \xi_s + \mathcal{O}(\epsilon^2). \tag{271}$$

Defining a diagonal matrix $\mathcal{D}_2$,

$$\mathcal{D}_2(m,m') = \mathbf{1}_{m=m'} \frac{\phi_m}{1-\phi_m} \frac{\partial U}{\partial C_m} \frac{\partial F_m}{\partial L_m} \left(1 - \eta_m \left(\frac{\partial F_m}{\partial L_m}\right)^{-2} \frac{\partial^2 F_m}{\partial L_m^2} \bar{C}_m\right),$$

and using a first-order approximation to the definition of the price index, (271) becomes

$$-A_{lc}\bar{C}_T c_{Ts} + A_{ll}(\bar{L} l_s) = \mathcal{D}_2 p_s + A_{l\xi} \cdot \xi_s + \mathcal{O}(\epsilon^2).$$

Rewriting in terms of the output gap and solving,

$$p_s = \kappa x_s + \mathcal{O}(\epsilon^2), \tag{272}$$

where $\kappa \in \mathbb{R}^{M \times M}$ is given by

$$\kappa = \mathcal{D}_2^{-1} A_{ll} \mathcal{D}_1^{-1}.$$

Replacing in the objective function,

$$U_s = \frac{\partial U}{\partial C_T} \bar{C}_T \left(c_{Ts} + \frac{1}{2} c_{Ts}^2\right) + \frac{1}{2} A_{\mathcal{T}\mathcal{T}} \bar{C}_T^2 c_{Ts}^2 + \bar{C}_T c_{Ts} \left(A_{c\xi} + A'_{lc} A_{ll}^{-1} A_{l\xi}\right) \xi_s - \frac{1}{2} x' \tilde{\Lambda}_x x + \text{t.i.p.} + \mathcal{O}(\epsilon^3)$$

where

$$\tilde{\Lambda}_x \equiv \kappa' \Lambda_p \kappa + \Lambda_x.$$

Using a second-order approximation of the budget constraint and the foreign no-arbitrage equations,

$$\mathcal{W} = \sum_s \pi_s \{-\frac{1}{2} A_{\mathcal{T}\mathcal{T}} \mathcal{T}_s^2 + \mathcal{T}_s A_{\mathcal{T}\xi} \cdot \xi_s - \frac{1}{2} x' \tilde{\Lambda}_x x\} + \text{t.i.p.} + \mathcal{O}(\epsilon^3) \tag{273}$$

where

$$A_{\mathcal{T}\xi} = A_{c\xi} + A'_{lc} A_{ll}^{-1} A_{l\xi} + \left(\frac{\partial^2 U}{\partial C_T^2} + A'_{lc} A_{ll}^{-1} A_{lc}\right) D_\xi Y_T + \frac{\partial U}{\partial C_T} \gamma^* D_\xi C^*.$$

Monetary policy Because prices are sticky, the exchange rate can affect the overall level of labor. Indeed, consumer optimization implies

$$U_{C_m}(s) = E_s^{-1} P_{ms} U_{C_T}(s)$$

A first-order approximation of this equation yields

$$\left(\Gamma_T + \Gamma_M A_{ll}^{-1} A_{lc}\right) \mathcal{T}_s + \left(\Gamma_\xi + \Gamma_M A_{ll}^{-1} A_{l\xi} + \left(\Gamma_T + \Gamma_M A_{ll}^{-1} A_{lc}\right) D_\xi Y_T\right) \cdot \xi_s + \left(\Gamma_M \mathcal{D}_1^{-1} - \kappa\right) x_s = \tag{274}$$

$$\mathbf{1}_{M\times 1} \left(-e_s - \left(\frac{\partial U}{\partial C_T}\right)^{-1} A_{\mathcal{T}\mathcal{T}} \mathcal{T}_s + \left(\left(\frac{\partial U}{\partial C_T}\right)^{-1} A_{\mathcal{T}\xi} - \gamma^* D_\xi C^*\right) \cdot \xi_s + \left(\frac{\partial U}{\partial C_T}\right)^{-1} A'_{lc} \mathcal{D}_1^{-1} x_s\right) + \mathcal{O}(\epsilon^2)$$

where $\Gamma_T \in \mathbb{R}^{M\times 1}$, $\Gamma_\xi \in \mathbb{R}^{M\times S}$, $\Gamma_M \in \mathbb{R}^{M\times M}$ are given by:

$$\Gamma_T(m) \equiv \left(\frac{\partial U}{\partial C_m}\right)^{-1} \frac{\partial^2 U}{\partial C_m \partial C_T}$$

$$\Gamma_\xi(m,:) \equiv \left(\frac{\partial U}{\partial C_m}\right)^{-1} \left(D_{C_m \xi} U + \sum_{m'} \frac{\partial^2 U}{\partial C_m \partial C_{m'}} D_\xi F_{m'}\right)$$

$$\Gamma_M(m,m') \equiv \left(\frac{\partial U}{\partial C_m}\right)^{-1} \left(\frac{\partial^2 U}{\partial L_m \partial C_{m'}} + \frac{\partial^2 U}{\partial C_m \partial C_{m'}} \frac{\partial F_{m'}}{\partial L_{m'}}\right).$$

Planner's problem and optimal tax The planning problem is to maximize (273) subject to (274), the definition of the transfer

$$\mathcal{T}_s = \sum_j \bar{\Theta}_j rr_{js} \tag{275}$$

and the definition of realized excess returns, given to first order by

$$k_{rr_j c} \mathcal{T}_s + k_{rr_j e} e_s + k_{rr_j x} x_s + k_{rr_j \xi} \xi_s = rr_{js}, \tag{276}$$

where $k_{rr_j c}, k_{rr_j e} \in \mathbb{R}$, $k_{rr_j x} \in \mathbb{R}^{1\times M}$, $k_{rr_j \xi} \in \mathbb{R}^{1\times S}$.

Let ν_s denote the multiplier on (274) and substitute (276) into (275) and let λ_s denote the multiplier on the resulting constraint. The FOC of this problem are

$$-A_{\mathcal{T}\mathcal{T}}\mathcal{T}_s + A_{\mathcal{T}\xi}\cdot\xi_s + \nu_s'\tilde{\Gamma}_T - \left(1 - \sum_j \bar{\Theta}_j k_{rr_j c}\right)\lambda_s = 0$$
$$\nu_s'\mathbf{1}_{M\times 1} + \sum_j \bar{\Theta}_j k_{rr_j e}\lambda_s = 0$$
$$-\tilde{\Lambda}_x x_s + \sum_j \bar{\Theta}_j k_{rr_j x}'\lambda_s + \tilde{\Gamma}_M'\nu_s = 0$$
$$\sum_s \pi_s rr_{js}\lambda_s = 0.$$

where

$$\tilde{\Gamma}_T = \Gamma_T + \Gamma_M A_{ll}^{-1}A_{lc} + \mathbf{1}_{M\times 1}\left(\frac{\partial U}{\partial C_T}\right)^{-1} A_{\mathcal{T}\mathcal{T}}$$
$$\tilde{\Gamma}_M = \Gamma_M\mathcal{D}_1^{-1} - \kappa - \mathbf{1}_{M\times 1}\left(\frac{\partial U}{\partial C_T}\right)^{-1} A_{lc}'\mathcal{D}_1^{-1}$$

Combining the last FOC with the second FOC,

$$\sum_s \pi_s rr_{js}\nu_s'\mathbf{1}_{M\times 1} = 0.$$

Solving for ν_s in the third equation and replacing,

$$\sum_s \pi_s \omega rr_{js} x_s = 0$$

where

$$\omega = -\mathbf{1}_{1\times M}\left(\tilde{\Gamma}_M'\right)^{-1}\tilde{\Lambda}_x \in \mathbb{R}^{1\times M} \tag{277}$$

is a $1\times M$ vector of weights. Intuitively, the planner only allows "average" booms and recessions to improve insurance. ω controls how important each of these output gaps are for welfare.

To study the optimal tax, I combine a second-order approximation of home and foreign no-arbitrage conditions,

$$\sum_s \pi_s\left(-A_{\mathcal{T}\mathcal{T}}\mathcal{T}_s + A_{lc}'x_s + A_{\mathcal{T}\xi}\cdot\xi_s\right) rr_{js} = \frac{\partial U}{\partial C_T}\tau_j$$

Combining this with the first FOC of the planner's problem and portfolio optimality (the last FOC),

$$\sum_s \pi_s\left(\mathcal{A}_x x_s rr_{js}\right) = \frac{\partial U}{\partial C_T}\tau.$$

where

$$\mathcal{A}_x = -\left(\tilde{\Gamma}_T\right)'\left(\tilde{\Gamma}_M'\right)^{-1}\tilde{\Lambda}_x + A_{lc}' \in \mathbb{R}^{1\times M} \tag{278}$$

As one would expect, a tax is needed if the returns of the assets are correlated with the output gaps. Crucially, the weights that matter for the wedge between social and private marginal utility

$\mathcal{A}_x$ are not typically the same as the one that the planner seeks to stabilize for demand-management reasons, ω. Intuitively, if good j has a high $\mathcal{A}_x(j)$ relative to $\omega(j)$, then the economy will feature significant booms and recessions in good j that matter for the wedge but matter little for welfare. Thus, ceteris paribus, assets with a high return when good j is booming will be overvalued and vice versa. There are a few remarkable cases, however, where $\mathcal{A}_x$ and ω are proportional to one another: separable utility and GHH preferences (the latter only with rigid prices).

I collect these results in the following proposition.

Proposition 23. *In a multi-sector small open economy model (i.e. $m \to \infty$), the optimal policy prescribes that a weighted average of the output gaps is uncorrelated with the returns of available assets,*

$$\sum_s \pi_s \omega rr_{js} x_s = \mathcal{O}(\epsilon^3).$$

where ω is given by (277). Furthermore, the optimal taxes are given by

$$\sum_s \pi_s \left(\mathcal{A}_x x_s rr_{js}\right) = U_{C_T}\tau + \mathcal{O}(\epsilon^3),$$

where $\mathcal{A}_x$ is given by (278). Therefore, if there exists a constant $\mathcal{K} \in \mathbb{R}$ such that

$$\omega = \mathcal{K}\mathcal{A}_x,$$

then the optimal tax is zero. Examples of economies where such a constant exists are:
(i) Models with symmetric nontradable sectors;
(i) A model where tradables are separable from nontradables, i.e.

$$U(C_{Ts}, \{C_{ms}\}_m, \{L_{ms}\}_m) = U_T(C_{Ts}) + U_N(\{C_{ms}\}_m, \{L_{ms}\}_m);$$

(ii) A model with rigid prices ($\phi_m = 1 \ \forall m$) and GHH preferences

$$U(C_{Ts}, \{C_{ms}\}_m, \{L_{ms}\}_m) = \frac{1}{1-\gamma}\left(\left(\alpha_T^{\frac{1}{\rho}} C_{Ts}^{\frac{\rho-1}{\rho}} + \sum_{m=1}^{M} \alpha_m^{\frac{1}{\rho}} C_{ms}^{\frac{\rho-1}{\rho}}\right)^{\frac{\rho}{\rho-1}} - \sum_{m=1}^{M} \frac{\alpha_m}{1+\varphi_m} L_{ms}^{1+\varphi_m}\right)^{\frac{1}{1-\gamma}},$$

where $\alpha_T + \sum_{m=1}^{M} \alpha_m = 1$. I verified this analytically for $M = 2$, but numerical explorations suggest it works for an arbitrarily large $M \in \mathbb{N}$.

Proof. (i) is immediate from the fact that all rows will give identical results if all nontradable sectors are identical;[62] (ii) when tradables are separable from nontradables, $\Gamma^T = A_{lc} = 0_{M\times 1}$. Therefore,

$$\mathcal{A}_x = \underbrace{\left(\frac{\partial U}{\partial C_T}\right)^{-1} A_{\mathcal{T}\mathcal{T}}}_{\mathcal{K}} \underbrace{\left(-\mathbf{1}_{1\times M}\left(\tilde{\Gamma}'_M\right)^{-1}\tilde{\Lambda}_x\right)}_{\omega}.$$

(ii) After some algebra, one can show that when $M = 2$ and prices are fully rigid,

$$\mathcal{A}_x = \underbrace{\frac{1}{\rho\alpha_T}}_{\mathcal{K}}\omega.$$

[62] The symmetry here refers to how they enter into utility; shocks ξ can vary across sectors.

□

Examples of non-zero taxes A model with asymmetric wealth effects can produce a non-zero tax. Consider, for example, a model where one good is separable and the other is not:

$$U(C_T, C_1, C_2) = \ln\left(\alpha_T^{-\alpha_T}\alpha_1^{-\alpha_1}C_T^{\alpha_T}C_1^{\alpha_1} - \alpha_1 L_1\right) + \alpha_2 \ln C_2 - \alpha_2 L_2$$

and linear production $C_m = Z_m L_m \; \forall m$. Simulating this model with $\alpha_T = 0.4$, $\alpha_1 = \alpha_2 = 0.3$, and rigid prices ($\phi_1 = \phi_2 = 1$), I find that $\mathcal{A}_x$ is relatively larger than ω for the second (separable) good. For simplicity, consider an environment where $\bar{\Theta} = 0$ so that the solution features no transfers. Suppose that nontradable productivity of good 2 goes up. In this case, the planner reacts with a boom in sector 1 and a recession in sector 2. Since the externality places a higher weight on sector 2, the private sector undervalues consumption in this state. The opposite is true when z_1 and y are high (a y shock implies labor would ideally go up in sector 1 and stay constant in sector 2, so the former will have a recession and the latter a boom). In sum, assets that pay relatively more when z_2 is high are undervalued while assets that pay when z_1 is high are overvalued. Note that the case of a home-currency bond is non-obvious since the exchange rate moves in the same direction with both productivity shocks, i.e. z_1 and z_2.

A GHH model with heterogenous degrees of price stickiness can also give rise to a non-zero tax. Suppose utility is GHH with $\rho = \gamma = 1$, $\alpha_T = 0.4, \alpha_1 = \alpha_2 = 0.3$ and $\varphi_m = 0$.[63] Furthermore, suppose $\phi_1 = 0.5$ while $\phi_2 = 0.9$. In this model, per unit of output gap, both goods create the same wedge between private and social marginal utility. However, prices in sector 1 are more flexible and, hence, a large output gap implies substantial price dispersion, which is very costly. Hence, the planner allows for small output gaps in sector 1 and large ones in sector 2. As a result, assets that pay when sector 2 booms are overvalued, e.g. assets that pay when z_1 is large.

C Calibration details and additional numerical results

In this section, I present the calibration details omitted in the main text and I conduct additional exercises to shed additional light on the quantitative relevance of the insurance channel. In every case, I re-calibrate the volatility of the convenience-yield shock $\sigma(\psi)$ to match the observed portfolios under the Taylor rule.

C.1 Calibration details

I adopt standard values for the discount factor (0.99), risk aversion (2), and the Frisch elasticity of labor supply ($\frac{1}{2}$). I set the elasticity of substitution between tradable and nontradable goods at $\rho = 0.74$, following Mendoza (1992), who estimates it in a sample of 13 industrial countries. I assume that intermediate good producers do not reoptimize each period with probability 0.75, and set the elasticity across varieties η to 6, as in Gali and Monacelli (2005). For the remaining parameters, I use data from Canada, which I take as a benchmark small open economy. I classify as nontradable sectors those with a very low export share: construction and services related to real estate services, public administration, education, health services and professional and scientific services. This leads to a share of tradables in output (α) of 55%. Furthermore, I assume that the net foreign asset position is balanced (i.e, $NFA_{ss} = 0$), which is roughly in line with the average NFA in Canada over the past decade. The coefficients of the Taylor rule are borrowed from Verstraete and Suchanek (2018), who estimate the coefficients of the Taylor rule for Canada (table 4a in their

[63] One can also get a non-zero tax with heterogeneity in φ_m or F_m provided $\phi_m < 1$ or F_m is nonlinear in L_m.

paper).[64] Since I lack data on the maturity of home-currency external debt, I choose δ to match an average maturity of 6 years, which roughly corresponds to the average maturity of Canadian government debt.[65]

C.2 Nominal rigidities

In this section, I study the sensitivity of the results to changes in the parameters that govern nominal rigidities. First, I assume that nontradable producers reset their prices more often, reducing ϕ from 0.75 to 0.7. Second, I assume a higher elasticity of substitution across varieties, $\eta = 11$. Finally, I assume a lower elasticity of labor supply, $\varphi = 10$.

Table 4 shows the results. As one may see, the size of the gross positions and the relative importance of the insurance target increase substantially when prices are more flexible.[66] Indeed, under the optimal policy both objectives become equally important. Like before, portfolio endogeneity is crucial: if the portfolio were not able to adjust, the demand-management objective would still be by far the most important goal of monetary policy. As one may expect, the planner can now reap more benefits out of financial integration. An increase in the elasticity of subtitution has a similar effect in the opposite direction.

A decrease in the labor supply elasticity has ambiguous effects. On the one hand, it decreases the exogenous parameter that controls the importance of the insurance motive, χ: a given output gap is more costly if the disutility of labor is more convex (see section 2.3 for an explicit expression in the static model). It also strenghtens the wealth effect discussed in section 3.6, which also decreases the importance of the insurance motive because short home-currency positions become more costly. On the other hand, φ also lowers the volatility of the realized returns under demand-management, implying that large gross positions will not typically create large undesired transfers of wealth after e.g., productivity shocks. In this numerical example, both forces offset each other so that the final outcome is similar to the baseline model.

C.3 Finite number of foreign arbitrageurs

Next, I consider a deviation from the small open economy assumption that I studied in section 4: only a mass m of foreigners may access Canadian home-currency bond markets (the risk-free bond in foreign currency is still in infinitely elastic supply at R^*). I consider different m to vary the amount of total wealth that the foreign arbitrageurs have. Since a finite number of foreigners introduces a terms-of-trade manipulation effect, I report the optimal tax on home-currency bonds and consider a fifth policy: the solution under cooperation.

[64] Note that the coefficients that they report for inflation and output gaps correspond to $(1-\rho_i)\phi_\pi$ and $(1-\rho_i)\phi_x$ in my model, respectively.

[65] Bank of Canada reports government debt by maturity grouped into time brackets: up to 3 months, from 3 months to 3 years, from 3 to 5 years, from 5 to 10 years, and over 10 years (https://www.bankofcanada.ca/rates/banking-and-financial-statistics/government-of-canada-direct-securities-and-loans-classified-by-remaining-term-to-maturity-and-type-of-asset-formerly-g6/). I compute average maturity as a weighted mean of the average and the maximum value of the brackets, which takes values between 5.7 and 6.7 for the years in the sample.

[66] One reason why these quantitative results are fairly sensitive to the value of these parameters is that the correlation between both exchange-rate targets is not very large in the calibrated model (see table 2). One may see from the solution to the optimal portfolio problem (see equation (97)) that, if the correlation were exactly zero, the solution would be "bang-bang": pick $\bar{B} = 0$ if demand-management is more important, and $\bar{B}/(1-\mu\bar{B}) \to \pm\infty$ if insurance is. In other words, there is a threshold value χ^* such that the behavior of the optimal portfolio varies drastically around it. Similarly, in models where the correlation is small, there exists a region of the parameter space where optimal portfolios become very sensitive to the values of parameters that affect χ, such as η and ϕ.

Table 5 shows the results. As the number of foreigner arbitrageurs decreases, the size of optimal positions naturally decreases: it is more expensive for foreigners to hold the home-currency bond because it induces additional volatility in their consumption. Importantly, the home planner understands that, as the country issues more home-currency debt, the yield on this debt increases. Thus, it taxes home-currency debt (subsidize home-currency assets) to induce agents to reduce their home-currency debt against the rest of the world. The tax is substantial; slightly under 100% of the expected excess returns under demand-management and over 100% under the optimal policy.[67] Interestingly, when there are very few foreigners (bottom panel) the tax is so large that under the optimal policy the observed level of financial integration is actually smaller than under laissez-faire with a Taylor rule. Furthermore, note that the insurance weight increases as m decreases when the portfolio is fixed (column 4). This is because the planner starts caring about transfers of wealth not only because of their effect on risk sharing, but also on the price of the bond. Indeed, one can see from equation (256) that χ decreases with m. Similarly, because this terms-of-trade manipulation motive exists even under complete markets and flexible prices, there is not much financial integration even under the first best with non-cooperative policy. As a result, the demand-management and optimal policies attain a larger share of the potential gains of trading the home-currency bond with the rest of the world.

Finally, note that the limits to arbitrage in home-currency bond markets imply that there are substantial gains for the world of cooperating. When the foreign welfare is taken into account, the planner once again chooses large gross positions under the optimal policy. In addition, note that the welfare gains of financial integration relative to the first-best decrease with m. The reason for this is that the model is being recalibrated as m changes to match positions under the Taylor rule. Since a smaller m naturally makes larger positions more expensive, the required volatility of the convenience-yield shock also decreases. As a result, the correlation between both exchange-rate targets increases and the planner gets closer to the first-best without distorting demand.

C.4 Savings taxes and bond duration

In section 5.3, I emphasized that in a dynamic model the planner does not only rely on monetary policy but also on savings taxes (i.e., taxes on cross-border flows that are uniform across assets) to manipulate the realized return of the home-currency bond. Here, I compute the savings taxes that are implied by the model under the optimal policy and compare the solution to the case where these taxes are unavailable (see appendix B.4.15 for the theoretical derivation). In that section, I also emphasized that the optimal savings taxes crucially depends on the maturity of the bond. To study this, I consider a case with a 1-year bond ($\delta = \frac{1}{4}$) and a case with a 10-year bond ($\delta = \frac{1}{40}$).

Table 6 shows the results. In the baseline calibration, taxes allow the planner to increase the weight on the insurance motive and reap more benefits from financial integration. However, the effects are not very large: without taxes the optimal weight only decreases by two percentage points while the gross position decreases by 7 percentage points of GDP. Taxes are rather small: their standard deviation is only 0.07%. These taxes become an order of magnitude larger when considering one-year bonds. Accordingly, the planner can provide much more insurance when they are available. The intuition is given by proposition 14. When bonds are short, they promise a stream of payments with a larger variability over time, i.e. a large payment today and small tomorrow. As a result, manipulating the path of tradable consumption and, hence, of the real exchange rate that closes the output gap, is very effective at manipulating the payments of these bonds. By contrast, when they are long, they promise similar payments over time and, hence, distorting the path of tradable consumption is less effective at distorting the value of the payment stream.

[67] Recall that this tax, like the expected excess returns (i.e., the risk premium), are second-order objects.

Finally, comparing results across duration, one may see that the planner can provide more insurance when bonds are long. The main reason is that the parameter that governs the importance of insurance χ increases with bond maturity. When bonds are long, the planner can make exchange rate promises far into the future. These promises affect the value of the long bonds today, but create little production distortions: these movements are expected and give firms time to adjust their prices.

C.5 Other parameters

In the main text, I set $\alpha = 0.55$ to reflect the share of the tradable sector in Canada. However, as argued by Burstein, Neves and Rebelo (2003), tradable goods have a large distribution cost component, which is also nontradable. For Canada, they estimate this distribution cost to be 40%. Making this correction implies a tradable share of $\alpha = 0.33$. Since the nontradable sector is the one affected by the nominal rigidity in my model, this makes demand-management more important. Accordingly, the planner reduces the insurance weight to 6.33% and lowers the optimal position in home-currency debt to 59%.

Next, I study the role of the complementarity between tradable and nontradable goods. I consider two values, which correspond to the bounds on the estimates in the literature (see Akinci (2011) for a survey): $\rho = 0.4$ and $\rho = 1.5$. A lower elasticity of substitution decreases the pass-through of the exchange rate to the output gap, which lowers the cost of providing insurance. In addition, it makes capital controls more effective: the wealth effect becomes more important (i.e., μ is larger) and, thus, changes in tradable consumption have a larger effect on the exchange rate that closes the output gap.[68] Overall, the effects are significant: at the lower end, the optimal weight on insurance increases by five percentage points, while it decreases by two percentage points at the upper end of the admissible values for ρ.

Next, I vary risk aversion (column 5). I set $\gamma = 10$ - the upper bound of the range considered by Mehra and Prescott (1985). A higher risk aversion naturally makes insurance more important. Thus, the optimal insurance weight increases, gross positions become larger, and there are larger gains of financial integration.

Finally, I change the discount factor (column 6). For illustrative purposes, I set β=0.98, which is very low for a model at the quarterly frequency. Ceteris paribus the shocks, a higher discount factor implies transfers become more valuable. It has a similar effect to risk aversion, although its effects are more modest.

[68] When $\rho = 0.4$, $\sigma(\tau_{sav}) = 0.16\%$. When $\rho = 1.5$, $\sigma(\tau_{sav}) = 0.04\%$.

Table 4: Varying the importance of demand-management

	Taylor rule	Demand management	Optimal	Optimal: fixed Θ
A. Benchmark				
ω		0%	11.37%	1.66%
$\bar{\Theta}$	−30.00%	−28.97%	−85.83%	−30.00%
Welfare gains	1.32%	2.28%	6.49%	3.90%
B. Low price stickiness ($\phi = 0.7$)				
ω		0%	54.52%	2.46%
$\bar{\Theta}$	−30.00%	−28.80%	−237.28%	−30.00%
Welfare gains	1.31%	2.26%	16.25%	4.66%
C. High elasticity of substitution across varieties ($\eta = 11$)				
ω		0%	2.29%	0.95%
$\bar{\Theta}$	−30.00%	−28.97%	−47.38%	−30.00%
Welfare gains	0.57%	2.28%	3.68%	3.21%
D. Low elasticity of labor supply ($\varphi = 10$)				
ω		0%	11.50%	1.64%
$\bar{\Theta}$	−30.00%	−29.67%	−88.14%	−30.00%
Welfare gains	−0.66%	2.36%	6.65%	3.96%

Note: Welfare gains are measured by how much of the welfare gap between the first-best (a model with flexible prices) and an economy without home bonds ($\bar{B} = 0$) economy is achieved by each policy: $\frac{welfare(policy) - welfare(\bar{B}=0)}{welfare(firstbest) - welfare(\bar{B}=0)}\%$. Note that the Taylor rule is not guaranteed to deliver positive welfare gains (an economy without home-currency bonds and flexible prices/perfect demand-targeting may dominate it).

Table 5: Finite number of arbitrageurs

	Taylor rule	Demand management	Optimal	Optimal: fixed Θ	Optimal: Cooperation
		A. Benchmark ($m \to \infty$)			
ω		0%	11.37%	1.66%	11.37%
$\bar{\Theta}$	−30.00%	−28.97%	−85.83%	−30.00%	−85.83%
τ_B/risk premium		0%	0%		0%
Welfare gains	1.32%	2.28%	6.49%	3.90%	6.49%
		B. Equal number of home and foreign agents ($m = 1$)			
ω		0%	6.19%	3.69%	15.77%
$\bar{\Theta}$	−30.00%	−20.84%	−39.63%	−30.00%	−80.88%
τ_B/risk premium		−74.17%	−176.1%		0%
Welfare gains	0.83%	3.73%	7.01%	6.63%	10.02%
		C. Very few foreigners ($m = \frac{1}{10}$)			
ω		0%	11.94%	18.78%	31.94%
$\bar{\Theta}$	−30.00%	−16.08%	−22.86%	−30.00%	−59.56%
τ_B/risk premium		−82.65%	−115.49%		0%
Welfare gains	−10.96%	17.43%	24.66%	22.78%	33.60%

Note: Welfare gains are measured by how much of the welfare gap between the first-best (a model with flexible prices) and an economy without home bonds ($\bar{B} = 0$) economy is achieved by each policy: $\frac{welfare(policy)-welfare(\bar{B}=0)}{welfare(firstbest)-welfare(\bar{B}=0)}\%$. The first-best is computed from the point of view of the home economy in the first four columns, and from the point of view of a global planner in the last column (i.e. even under complete markets and flexible prices, a non-cooperative planner would like to manipulate the stochastic discount factor of foreigners). Note that the Taylor rule is not guaranteed to deliver positive welfare gains (an economy without home-currency bonds and flexible prices/perfect demand-targeting may dominate it).

Table 6: Savings taxes and bond duration

	Optimal policy: with taxes	Optimal policy: without taxes
A. Benchmark ($\delta = \frac{1}{24}$)		
ω	11.37%	9.17%
$\bar{\Theta}$	−85.83%	−77.99%
$\sigma(\tau_{\text{sav.}})$	0.08%	0%
Welfare gains	6.49%	5.93%
B. One-year bonds ($\delta = \frac{1}{4}$)		
ω	6.01%	0.87%
$\bar{\Theta}$	−67.49%	−37.64%
$\sigma(\tau_{\text{sav.}})$	0.84%	0%
Welfare gains	4.59%	2.62%
C. Ten-year bonds ($\delta = \frac{1}{40}$)		
ω	50.35%	46.81%
$\bar{\Theta}$	−217.18%	−202.67%
$\sigma(\tau_{\text{sav.}})$	0.08%	0%
Welfare gains	15.34%	14.45%

Note: Welfare gains are measured by how much of the welfare gap between the first-best (a model with flexible prices) and an economy without home bonds ($\bar{B} = 0$) economy is achieved by each policy: $\frac{welfare(policy) - welfare(\bar{B}=0)}{welfare(firstbest) - welfare(\bar{B}=0)}\%$.

Table 7: Other parameters

	Taylor rule	Demand management	Optimal	Optimal: fixed Θ
		A. Benchmark		
ω		0%	11.37%	1.66%
$\bar{\Theta}$	−30.00%	−28.97%	−85.83%	−30.00%
Welfare gains	1.32%	2.28%	6.49%	3.90%
		B. Low openness ($\alpha = 0.33$)		
ω		0%	6.33%	1.85%
$\bar{\Theta}$	−30.00%	−29.86%	−58.99%	−30.00%
Welfare gains	0.49%	3.66%	6.96%	5.43%
		C. Low elasticity of substitution T/NT ($\rho = 0.4$)		
ω		0%	15.33%	1.96%
$\bar{\Theta}$	−30.00%	−30.29%	−96.50%	−30.00%
Welfare gains	1.00%	2.82%	8.38%	4.72%
		D. High elasticity of substitution T/NT ($\rho = 1.5$)		
ω		0%	9.36%	1.44%
$\bar{\Theta}$	−30.00%	−27.87%	−81.35%	−30.00%
Welfare gains	1.27%	1.84%	5.27%	3.25%
		E. High risk aversion ($\gamma = 10$)		
ω		0%	19.16%	6.09%
$\bar{\Theta}$	−30.00%	−25.71%	−58.53%	−30.00%
Welfare gains	1.59%	7.29%	16.21%	12.75%
		F. High discount factor ($\beta = 0.98$)		
ω		0%	14.53%	3.03%
$\bar{\Theta}$	−30.00%	−28.74%	−74.36%	−30.00%
Welfare gains	2.30%	4.37%	10.62%	7.25%

Note: Welfare gains are measured by how much of the welfare gap between the first-best (a model with flexible prices) and an economy without home bonds ($\bar{B} = 0$) economy is achieved by each policy: $\frac{welfare(policy) - welfare(\bar{B}=0)}{welfare(firstbest) - welfare(\bar{B}=0)}\%$.